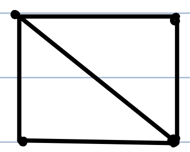
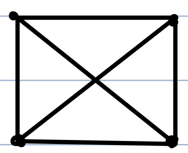
Planar graphs

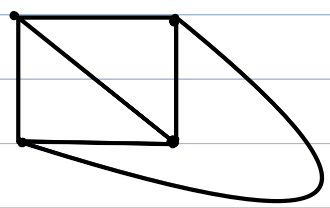
Definition: A graph is planar if it has a drawing (on the plane $\mathbb{R}^2$) without any edges crossing (they are allowed to meet at a common vertex). A planar drawing or a planar representation of a graph is such a drawing of the graph.

Examples

1.  Clearly planar.

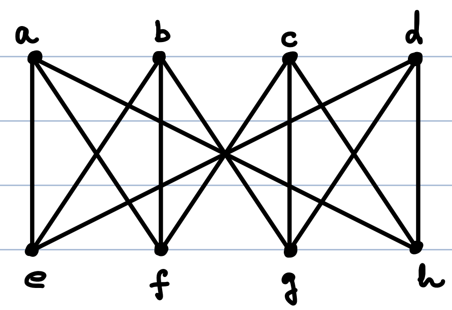
2.  Looks non planar. However it is planar. See

This graph is usually denoted K_4. This means it has 4 vertices and an edge between any two distinct edges.

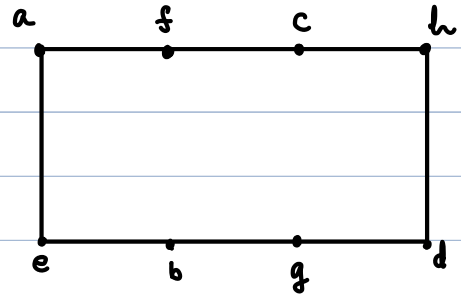


Another representation of the same graph. It is clear it is planar.

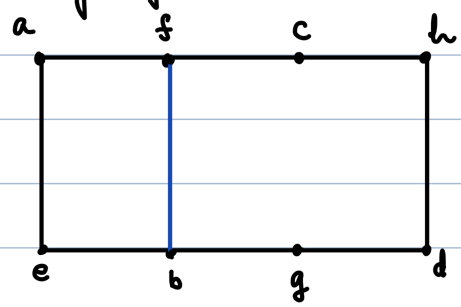
3. Consider the graph G, one of whose representations is:



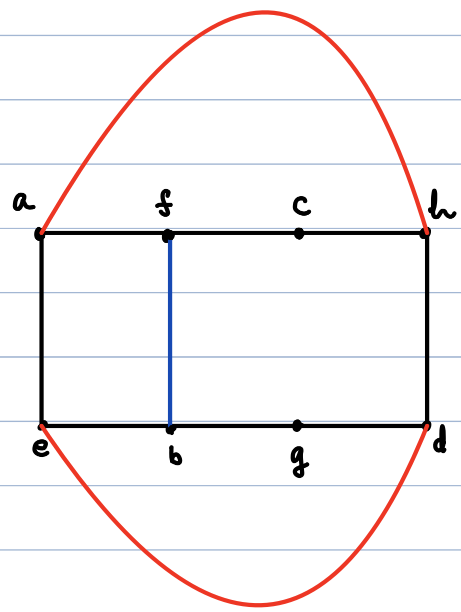
We can find a hamiltonian cycle in G, namely
$$\sigma = (a, f, c, h, d, g, b, e)$$



Next draw the edge fb as below

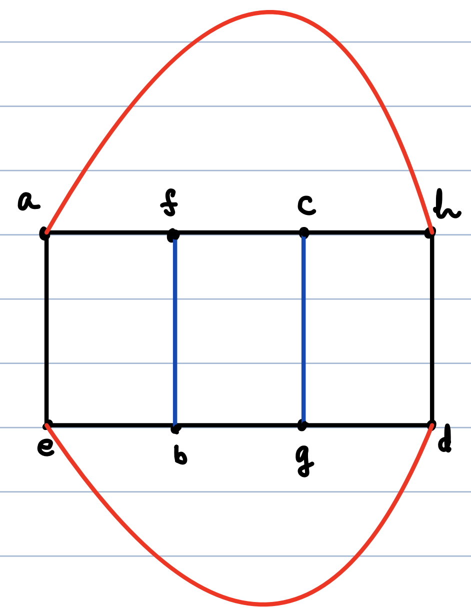


(The edge fb is often called a "chord" of the cycle). If you wish to draw the edges ah and ed so that they don't cross any of the edges in the drawing above you have no choice but the following:



This forces us to draw cg in the following way to keep edges from crossing:

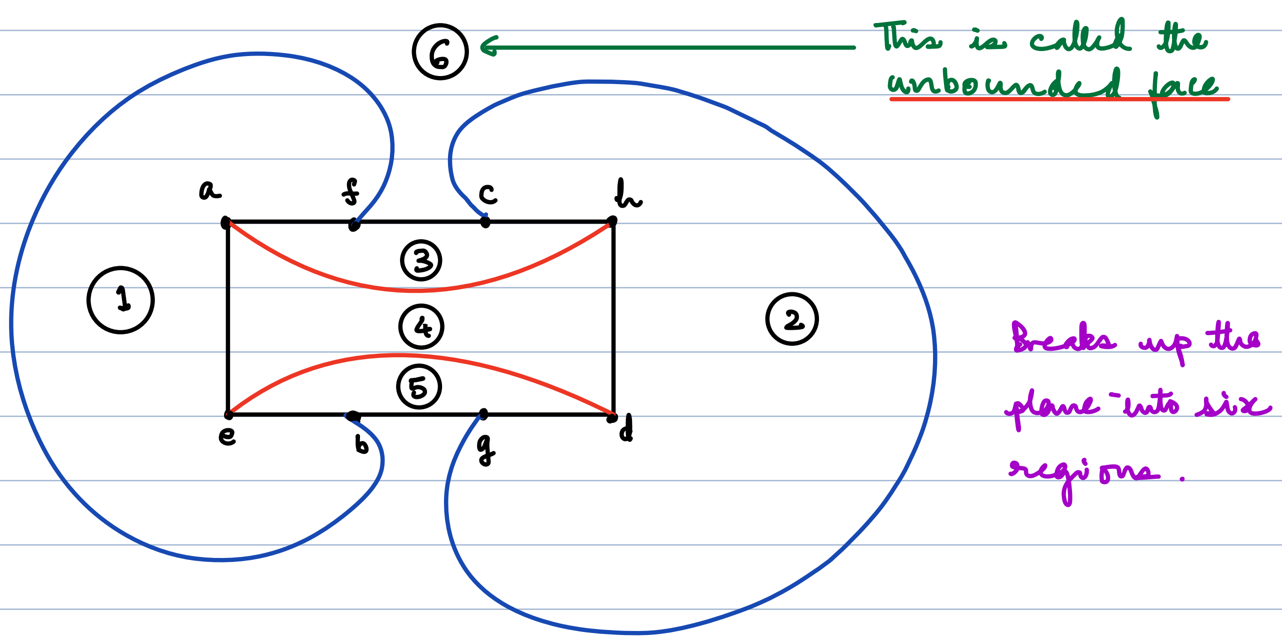
(See next page)



This shows that G is planar, something not obvious from the original drawing of G.

The above is often called the circle-chord method. We may say more later in the course.

Below is another representation of the same graph as a planar graph.



Take away: Planar representations are not unique.

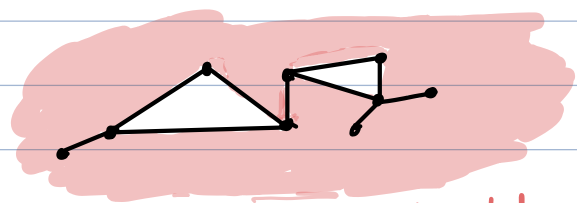
The above planar drawing breaks up the plane $\mathbb{R}^2$ into a number of region (six actually). Note that the region labelled 6 is not enclosed by edges. It is called the "unbounded region" or the "unbounded face" of the planar drawing.

Notations and terminology

Suppose $G=(V,E)$ is a planar graph. Represent it on the plane without any edges crossing. (There may be more than one representation as we just saw.) The planar graph's edges break up the plane into disjoint regions with the edges forming the boundaries of each region. In the picture above we have six regions. The regions are sometimes called faces of the graph. Let

$$f = \# \text{ of faces in planar drawing of } G.$$

All faces, except one, are enclosed by edges. The one that is not enclosed is called the unbounded face of the planar drawing of G, and the remaining are called the bounded faces of this planar drawing.



shaded region = unbounded component

Let

$$b = \# \text{ of bounded faces}.$$

Then, clearly

$$\boxed{f = b+1}$$

It may seem that f will depend upon the particular drawing of G as a planar graph. However, in the last example, note that $f=6$ in both planar representations of G. In fact, if G is connected, $f = 2+|E|-|V|$, a famous result of Euler, and hence is independent of the planar representation of G.

Euler's theorem is also important in topology: It is a way saying that the "Euler characteristic" of a sphere is -2. Read this up in a topology book if you have time. It is not needed in this course.

Theorem (Euler): Let $G=(V,E)$ be a connected planar graph with m edges and n vertices. Every planar drawing of G has f faces, where f satisfies

$$n-m+f=2.$$

Proof: We fix a planar drawing of G and let f be the number of faces. We prove the result by induction on m. If $m=0$, then $n=1$, since G is connected. It is clear, in this case, that $f=1$. Since $1-0+1=2$, the base case is verified.

Suppose now that G has m edges where $m \geq 1$, and that Euler's formula is true for all graphs with $m-1$ or fewer edges.

Case 1: G has a bounded face F. In this case F has a cycle σ in its boundary. Let e be an edge in σ. Form the graph $G'=(V',E')$ where $V'=V$ and $E'=E-\{e\}$. Then G is connected. Let $n'=|V'|$, $m'=|E'|$ and f' the number of faces in G'. Since $m'=m-1$, the induction hypothesis applies and hence

$$n'-m'+f'=2.$$

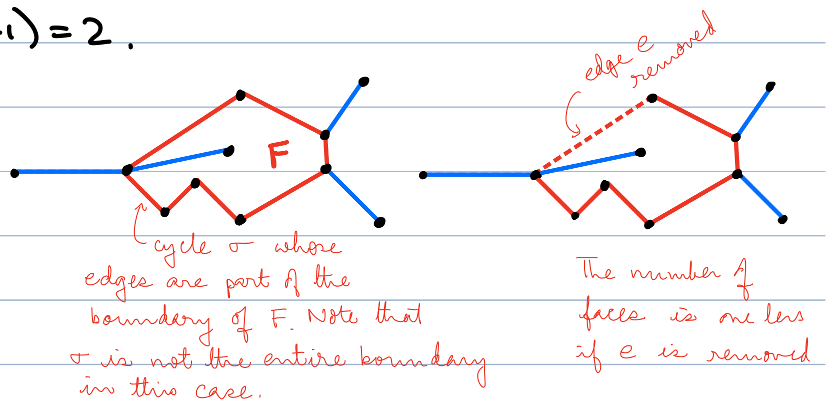
Note that since $V'=V$, therefore $n'=n$. Also $m'=m-1$. Also note that $f'=f-1$. Thus $n'-m'+f'=2$ translates to

$$n-(m-1)+(f-1)=2.$$

Thus

$$n-m+f=2$$

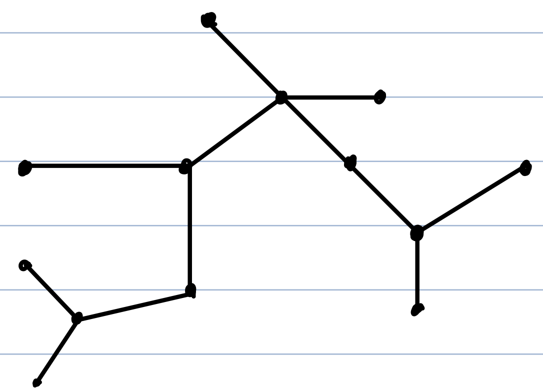
as required.



Case 2: G has no bounded face. This means $f=1$. If every vertex has two (or more) edges incident on it, then G has a cycle, which means it has a bounded face. Therefore G has a vertex v, with exactly one edge incident on it. Call this edge e. Consider the graph

$$G' = (V', E')$$

where $V' = V \setminus \{v\}$ and $E' = E - \{e\}$. Then G' is connected, and has no bounded faces. Let $n' = |V'|$, $m' = |E'|$, and f' the number of faces of the planar drawing of G'. Since G' has no bounded faces, $f' = 1$. Also $n' = n-1$, and $m' = m-1$. Since $m' < m$, our induction hypothesis applies and so $n' - m' + f' = 2$. Since $f = f' = 1$, this gives $(n-1) - (m-1) + f = 2$, i.e. $n - m + f = 2$, as required. //



A graph with NO bounded faces
It must have at least one vertex of degree 1.

Remark: If G is s.t. $\deg(v) \geq 2$ for every $v \in V$, then we can form a cycle as follows. Start with a vertex $x = x_1$. Pick x_2 adjacent to x_1. Since $\deg x_2 \geq 2$, we have x_3 adjacent to x_2, with $x_3 \neq x_1$. Next we can find x_4 adjacent to x_3 with $x_4 \neq x_2$. Either $x_4 = x_1$, in which case we stop, or we pick a neighbour x_5 of x_4 different from x_3. The process terminates whenever we revisit a vertex already visited. Since G is finite, the process has to terminate. Thus G must have a cycle!

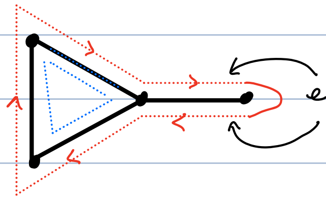
In other words, if G has no cycles, then there is a vertex of degree 1. In particular if G has a planar drawing with no bounded faces, then G must have vertex which has only one edge incident on it.

The degree of a face

Let F be a face of a planar drawing of a planar graph $G = (V, E)$. Set

$$\deg(F) := \# \text{ of edges on the boundary of } F.$$

The convention is that if an both sides of an edge e bound F, then e is counted twice. The reason for the convention is that as you go around the boundary of F you trace e twice as in the following picture.

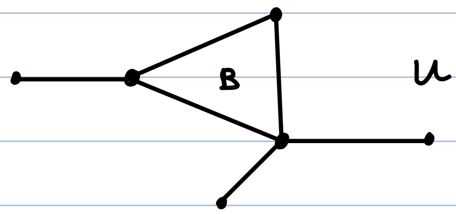


edge counted twice.

$\deg(F) = 5$ where F = unbounded face.

The number $\deg(F)$ is called the degree of F.

Example: Consider the graph below.



It has exactly two faces, a bounded one B, and an unbounded one U. Clearly

$$\deg B = 3$$
$$\deg U = 9.$$

Note that $\deg B + \deg U = 12 = 2(|E|)$

Theorem: Let $G = (V, E)$ be a planar graph, $R_1, \ldots, R_f$ the distinct faces of a planar drawing of G. Then

$$\sum_{i=1}^{f} \deg(R_i) = 2|E|.$$

Proof: This is clear since every edge is counted twice when adding the degrees of the faces. //

Remark: Note the similarity with the statement

$$\sum_{v \in V} \deg(v) = 2|E|.$$

If we wish to emphasize the role of G, we will write $\deg_G(F)$ instead of $\deg R$.

A necessary condition for a graph to be planar

Suppose $G=(V,E)$ is planar. Let $R_1,\dots,R_f$ be its distinct faces in a planar drawing of G, and $m_i = \deg R_i$. Let $m=|E|$, $n=|V|$. Suppose $n \geq 3$. Then it is clear that each R_i, including the unbounded face, has degree at least 3 (for bounded faces, this is obvious. It follows that

$$2m = \sum_{i=1}^{f} m_i \geq \sum_{i=1}^{f} 3 = 3f.$$

Assume G is connected.

By Euler's formula we have

$$f = m-n+2$$

The inequality $2m \geq 3f$ we just proved gives

$$2m \geq 3(m-n+2) = 3m-3n+6.$$

i.e.

$$3n-6 \geq m.$$

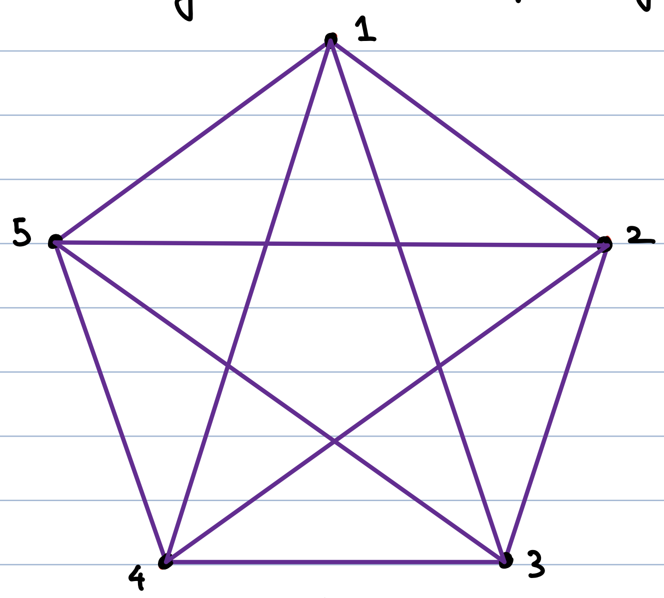
This gives us the following theorem.

Theorem: Let $G=(V,E)$ be a connected planar graph with $n=|V|$, $m=|E|$. If $n\geq 3$, then $3n-6\geq m$.

Definition: A graph $G=(V,E)$ is complete if any two distinct vertices are adjacent.

Note: It is obvious that if $G=(V,E)$ and $H=(W,F)$ are complete graphs with $|V|=|W|$, then G and H are isomorphic. The complete graph with vertices $1,2,\dots,n$ is denoted K_n, i.e. $K_n = ([n], \{\{i,j\} \mid i,j\in[n], i\neq j\})$. (See also problem 1 of the "Problems worth thinking about" section in the plans for week 6). If $G=(V,E)$ is complete, and $n=|V|$, then G is isomorphic to K_n. K_n is clearly connected & $\sigma=(1,2,\dots,n)$ is a hamiltonian cycle in K_n. Thus every complete graph is connected and hamiltonian.

Here is a drawing on the complete graph K_5.



K_5

Corollary: K_5 is not planar.

Proof:

We have $n = 5$, $m = 10$ and hence $3n - 6 = 9 < 10 = m$. //

Theorem: Every planar graph has a vertex of degree 5 or less.

Proof:

Without loss of generality, we may assume our graph is connected. Let $G = (V, E)$ be a connected planar graph, and set $n = |V|$, $m = |E|$. Suppose $\deg(v) > 5$ for every $v \in V$. This means $\deg(v) \geq 6$, $\forall\, v \in V$. Hence

$$2m = \sum_{v \in V} \deg(v) \geq \sum_{v \in V} 6 = 6 \cdot |V| = 6n.$$

Thus

$$m \geq 3n.$$

It follows that

$$m > m - 6 \geq 3n - 6$$

which, by the previous theorem, is impossible. //

Remark: The above is a necessary condition for a graph to be planar, not a sufficient condition. Indeed, every vertex in K_5 has degree 4, but, as we saw above, K_5 is not planar.

Graph colouring: A colouring of a graph is the assignment of a colour to each vertex of the graph so that no two adjacent vertices have the same colour.

A graph can be coloured by assigning a different colour to each of its vertices. However, for most graphs a colouring can be found that uses fewer colours than that. What is the least number of colours necessary?

Definition: The chromatic number of a graph is the least number of colours needed for a colouring of this graph.

Notation: $\chi(G)$ = chromatic number of graph G.

Theorem (The Four Colour Theorem): If G is planar then $\chi(G) \leq 4$.

The theorem was conjectured in the 1850s. It was an open problem for over 120 years when it was finally proven by Appel and Haken in 1976. The proof involved examing approximately 2000 possible counterexamples, and eliminating them via an analysis carried out by a computer. It used up over 1000 hours of computer time. Needless to say, we won't be giving a proof. While there have been improvements on the original proof (in terms of the number of graphs to be checked etc), there is none that is not computer based.

Let $n \in \mathbb{N}$. We say that graph G is n-colourable if $n \geq \chi(G)$. Thus the Four Colour Theorem says that every planar graph is 4-colourable. Since we are not going to prove that, as a consolation we offer this:

Theorem: Every planar graph is 5-colourable.

Proof:

Let $G = (V, E)$ be a planar graph. Let $n = |V|$. We will prove the theorem by induction on n. If $n \leq 5$, the theorem is clear.

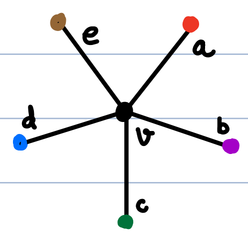
Suppose $n > 5$. Assume the theorem is true for any planar graph with fewer than n vertices. We know from an earlier result that since G is planar, G has a vertex v of degree less than or equal to 5.

Let H be the graph obtained from G by removing the vertex v (as well as all edges in G incident on v). By our induction hypothesis, H is 5-colourable, and so colour it with five or fewer colours.

If $\deg_G(v) \le 4$, then at most four colours have been used by the vertices adjacent to v (in the colouring of H) and so we can colour v by one of the colours not used up.

Suppose $\deg_G(v) = 5$. Let a, b, c, d, e be the vertices adjacent to v. If only four colours have been (when colouring H) for these five vertices, once again we can assign a colour to v so that G is 5-coloured.

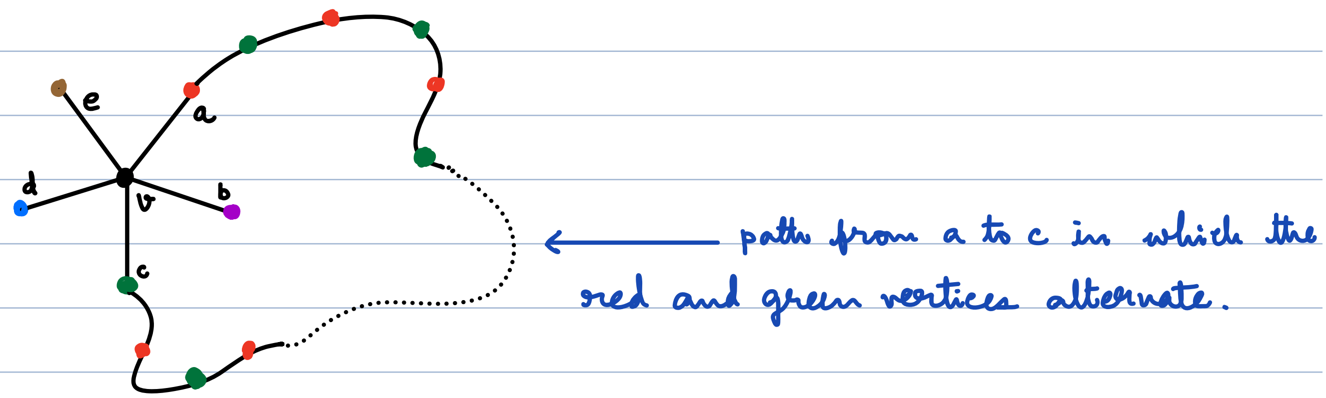
The only difficult case is when a, b, c, d, e, have been assigned five different colours, say as below, i.e. a is coloured red, b purple, c green, d blue, and e brown. Colour v black temporarily.



Let H_{RG} be the subgraph of H whose vertices are either red or green. In other words H_{RG} is obtained from H by removing all vertices coloured purple, blue, or brown, and all edges incident on such vertices.

Now consider the connected component H' of H_{RG} containing a. If H' does not contain c, then switch the two colours in H', so that the new colour of the red vertices in H' is green, and the new colour of the green vertices in H' is red. Since H' is a connected component of H_{RG}, H_{RG} remains 2-coloured in this new scheme, and H remains 5-coloured in this new scheme. However, now both a and c are green, and hence we can colour v red.

We could do this because c was not in H! What if c is a vertex in H'? In this case we have a path from a to c in H' (and hence in H) which consists of red and green vertices alternating with each other.



Let σ be the cycle obtained by adding the edge va to the front of this path and cv to the end.

There are two possibilities. Either b is inside σ and d is outside σ, or b is outside σ and d is inside. The situations are symmetric, and so we will assume b is inside σ and d is outside (as in the picture).

Consider any path τ in G from b to d. Since σ encloses b, τ must meet σ at some vertex. Since no vertex in σ is either purple or blue, this means τ cannot consist of only purple and blue vertices. This means that if H_{PB} is the sub-graph of H consisting of the purple and blue vertices in H, then the connected component of H_{PB} containing b does not contain d. Let H'' be this connected component. Switch the two colours in H'' (blues to purples and purples to blues). Then H remains 5-coloured. Now both b and d are blue. This means that if we colour v purple, then we have colouring of G with 5 colours as required. //