Notation. Let x be a vertex of a graph G. Then $N(x) = N_G(x)$ will denote the set vertices in G adjacent to x. In other words $N($ is the set of neighbours of x in G. $N(x)$ is called the neighbourhood of x in G.

Theorem (Dirac's theorem): Let $G=(V,E)$ be a connected graph with n vertices such that $n \geq 3$. If the degree of every vertex in G is at least $n/2$, then G is hamiltonian.

Proof:

Let $\sigma = (x_1, x_2, \dots, x_r)$ be a path in G of maximal length. This means if τ is another path in G then

$$\text{length}(\tau) \leq \text{length}(\sigma) = r-1.$$

Let X be the set of vertices in the path σ, i.e., $X = \{x_1, \dots, x_r\}$. Since σ is of maximal length $N(x_1) \subset X$ and $N(x_r) \subset X$, as we will see in a moment. Note that since $x \notin N(x)$, this means $N(x_1) \subset \{x_2, \dots, x_r\}$ and $N(x_r) \subset \{x_1, \dots, x_{r-1}\}$.

Let us now prove that $N(x_1)$ and $N(x_r)$ are subsets of X. Suppose $u \in N(x_1)$. If $u \notin X$, then

$$\tau = (u, x_1, \dots, x_r)$$

is path in G of length strictly greater than the length of σ, and this is not possible. Thus $u \in X$, i.e. $N(x_1) \subset X$. Similarly, if $u \in N(x_r)$ and $u \notin X$, then $(x_1, \dots, x_r, u)$ is a path longer than σ, which is not possible. So $u \in X$, i.e. $N(x_r) \subset X$.

Next we claim that $V = X$. To see this, suppose $\exists\ u \in V$ s.t. $u \notin X$.

Now $N(x_1) \subset \{x_2, \dots, x_r\}$

Let $S = \{x_j \in X \mid x_{j+1} \in N(x_1)\}$.

Then $|S| = |N(x_1)|$. Moreover $x_r \notin S$, by definition of S.

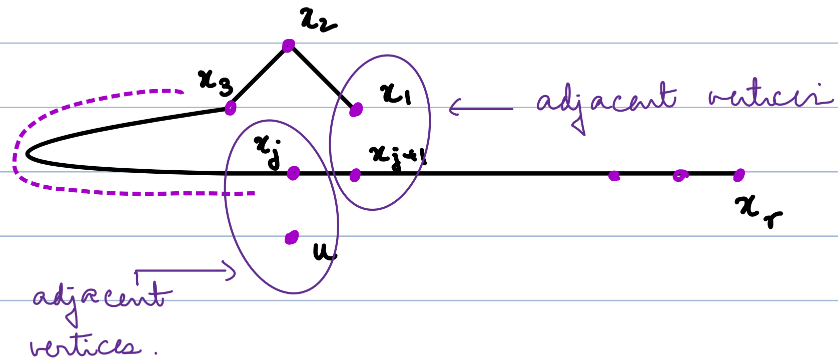
We claim $S \cap N(u) \neq \phi$. Suppose $S \cap N(u) = \phi$. Then S, $N(u)$, and $\{x_r\}$ are three disjoint subsets of V. Hence

$$|V| \geq |S| + |N(u)| + |\{x_r\}| \geq \frac{n}{2} + \frac{n}{2} + 1$$

i.e. $\quad n \geq n+1$.

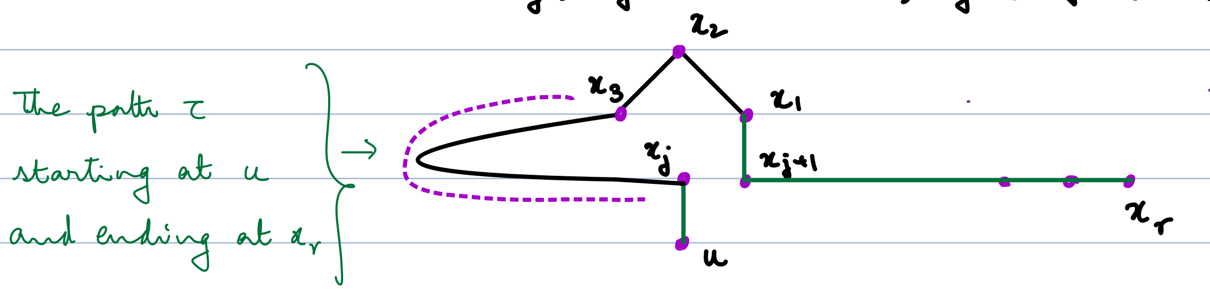
This is not possible. Hence $S \cap N(u) \neq \phi$.

Let $x_j \in S \cap N(u)$. Since $x_j \in S$, by definition of S, x_{j+1} is adjacent to x_1. Since $x_j \in N(u)$, x_j is adjacent to u.



Consider the path

$$\tau = (u, x_j, x_{j-1}, \ldots, x_2, x_1, x_{j+1}, x_{j+2}, \ldots, x_r)$$



The path τ is clearly longer than σ, giving a contradiction. Thus

$$V = X = \{x_1, \ldots, x_r\}.$$

Next we modify σ so that so that the modified path is a hamiltonian cycle.

As we observed, $N(x_1) \subset \{x_2, \ldots, x_r\}$. Similarly $N(x_r) \subset \{x_1, \ldots, x_{r-1}\}$ (we are using the fact that x_1 is not adjacent to itself and x_r is not adjacent to itself).

As before, let $S = \{x_j \in X \mid x_{j+1} \in N(x_1)\}$. We claim

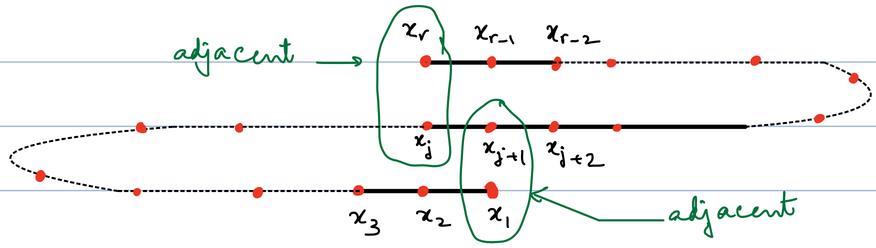
$$S \cap N(x_r) \neq \phi.$$

The argument is a repeat of what we had. Note that $x_r \notin S$ and $x_r \notin N(x_r)$. Suppose $S \cap N(x_r) = \phi$. Then

$$n = |V| \geq |S \cup N(x_r) \cup \{x_r\}| = |S| + |N(x_r)| + 1$$
$$\geq \frac{n}{2} + \frac{n}{2} + 1 = n+1$$

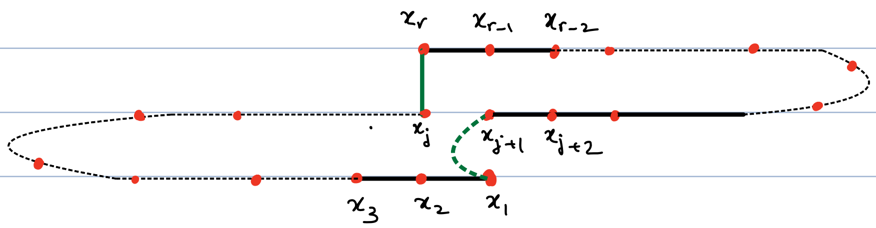
which is a contradiction.

Thus we have an element $x_j \in S \cap N(x_r)$. By definition of S, x_{j+1} is adjacent to x_1, and since $x_j \in N(x_r)$, x_j is adjacent to x_r.



Create a new path

$$\tau = (x_1, x_2, \ldots, x_j, x_r, x_{r-1}, x_{r-2}, \ldots, x_{j+2}, x_{j+1})$$



The path τ, starting at x_1 and ending at x_{j+1}. Note $x_{j+1}x_1$ is an edge.

Since x_{j+1} is adjacent to x_1 in G, τ is a cycle. In fact it is a hamiltonian cycle since it visits every vertex x_i, $i=1,\ldots,r$, and we proved earlier that $V = X = \{x_1, \ldots, x_r\}$.

This proves Dirac's theorem. //