Oct 17, 2022 Lecture 10 MAT344

Reminder: All graphs (unless otherwise stated) are assumed to be finite.

Recall that a graph is called eulerian if either it has only one vertex, or it has a circuit which traverses every edge exactly once. We had started the proof of Euler's theorem, namely: A graph is eulerian if and only if it is connected and the degree of every vertex is even.

We had proved one direction, namely, if a graph is eulerian then it is connected and every vertex has even degree.

Before we begin the proof of the converse, we need a definition.

Definition: Let $G=(V,E)$ be a graph. A trail in G is a walk $\sigma = (x_1, x_2, \ldots, x_n)$ such that the edges in the trail (i.e $x_i x_{i+1}$, $i=1,\ldots,n-1$) are distinct.

Definition: Let G be a graph. A vertex v of G is said to be isolated if no edge of G is incident on it.

Note: A vertex v is isolated if and only if $\deg(v)=0$.

Remark: It is easy to see that if $\sigma = (x_1, x_2, \ldots, x_{n+1})$ is a trail with $x_{n+1} = x_1$, then $n \geq 3$, and σ is a circuit. Conversely, clearly a circuit $\sigma = (x_1, \ldots, x_{n+1})$ is (by definition) a trail.

Building trails

Suppose v is a vertex in graph G with $\deg(v) > 0$. Then one can build a maximal trail starting at v as follows. Set $v = x_1$. Pick an edge $e_1 = vx_2 = x_1x_2$ incident on x_1. There is such an edge because $\deg(v) > 0$. If e_1 is the only edge incident to x_3 stop. If not, pick $e_2 = x_2x_3$ such that $e_2 \neq e_1$. Look at the edges incident to x_3. If there is any which is different from e_2, say x_3x_4, pick it and set $e_3 = x_3x_4$.

Suppose we have picked edges

$$e_1 = x_1x_2,\ e_2 = x_2x_3,\ e_3 = x_3x_4,\ \dots,\ e_i = x_ix_{i+1}$$

such that no edge equals any of the other edges. Look at all the edges incident to x_{i+1}. If there are none different from $e_1, e_2, \dots, e_i$, stop. If there is an edge $e = x_{i+1}y$ different from $e_1, e_2, \dots, e_i$, then set $x_{i+2} = y$ and $e_{i+1} = x_{i+1}x_{i+2}$. Since G is finite the process has to stop and we have a trail

$$\sigma = (\overset{v}{\overset{\shortparallel}{x_1}}, x_2, \dots, x_n)$$

such that all the edges incident to x_n are one of the $e_1, e_2, \dots, e_{n-1}$, where $e_j = x_jx_{j+1}$, $j = 1, \dots, n-1$. In other words vertices neighbouring x_n are a subset of $\{x_1, x_2, \dots, x_{n-1}\}$. One cannot expand the trail, since edges in trails are distinct.

Note: In the above trail if $x_n \neq x_1$, then the degree of x_n is odd, for the "incoming" edge $e_{n-1} = x_{n-1}x_n$ has no matching "outgoing" edge. The trail σ may visit x_n on earlier occasions, but when that happens, say $x_i = x_n$ for some $2 \le i \le n-2$, then the $e_{i-1} = x_{i-1}x_i = x_{i-1}x_n$ can be paired with $e_i = x_ix_{i+1} = x_nx_{i+1}$.

We have therefore proven the following lemma

Lemma: Let $G = (V, E)$ be a graph such that every vertex of G has even degree. If $v \in V$ has positive degree, then there is a circuit in G which begins and ends at v.

Proof:

Let $\sigma = (x_1, \dots, x_m)$ be a maximal trail (as constructed above) with $x_1 = v$. Since $\deg(x_m)$ is even, by the Note above, x_m must equal x_1. From the Remark above, σ is a circuit (set $(n = m-1$ if you wish, so that $m = n+1)$.

More notations: If $G=(V,E)$ is a graph and $v \in V$, we sometimes write $\deg_G(v)$ instead of $\deg(v)$ for the degree of v in G. This has advantage that if $H=(W,F)$ is a subgraph of G and $w \in W \subset V$, then we can distinguish between the degree of w in H and its degree in G. Clearly

$$\deg_H(w) \leq \deg_G(w).$$

Euler's Theorem

We restate the theorem

Theorem: A graph $G=(V,E)$ is eulerian if and only if it is connected and the degree of every vertex in G is an even number.

Proof: The case where $|V|=1$ is trivial and so we will assume from now on that $|V|>1$.

We have already shown (in the last lecture) that if G is eulerian then G is connected and the degree of every vertex in G is a positive even number.

Conversely, suppose G is connected and $\deg_G(v)$ is a positive even number for every $v \in V$. According to the Lemma above, there is a circuit in G. Let k be the largest possible length of a circuit in G. Since G is finite, k makes sense. Let

$$\sigma = (x_1, x_2, \ldots, x_{k+1})$$

be a circuit of length k, and $e_i = x_i x_{i+1}$, $i=1,\ldots,k$. Note that $x_{k+1}=x_1$.

Let $H=(W,F)$ be the subgraph of G given by.

$$W=V \quad \text{and} \quad F = E \smallsetminus \{e_1, e_2, \ldots, e_k\}.$$

In other words, H is obtained from G by removing all the edges of σ from G (the vertices are retained though).

Now H need not be connected. We claim that each of the vertices $x_1, x_2, \ldots, x_k$ of H is isolated in H. In other words, we claim that

Note: $\deg_H$ & not $\deg_G$ $\longrightarrow$ $\deg_H(x_i) = 0 \qquad i = 1, \ldots, k.$ ——— (*)

First note that $\deg_H(v)$ is an even number for every $v \in H$. Indeed, if $v \notin \{x_1, \ldots, x_k\}$, then $\deg_H(v) = \deg_G(v)$ and hence $\deg_H(v)$ is even (in fact positive and even). If on the other hand $v = x_i$ for some $i \in \{1, \ldots, k\}$, then the number of edges from the collection $\{e_1, e_2, \ldots, e_k\}$ incident on x_i is even, because every incoming edge is matched by an outgoing edge. Since

$$\deg_H(x_i) = \deg_G(x_i) - \#\text{ of edges in } \{e_1, \ldots, e_k\} \text{ incident to } x_i$$

it follows that $\deg_H(v) = \deg_H(x_i)$ is even.

We have to show $\deg_H(x_i) = 0$. Suppose not. By the Lemma there is a circuit σ' in H starting and ending at x_i. Putting together σ and σ' we get a circuit in G of length strictly larger than k. This contradicts the definition of k as the largest length of a circuit in G. Thus our claim (*) is true.

Next we claim that the edges $e_1, \ldots, e_k$ are all the edges in G, i.e. $E = \{e_1, e_2, \ldots, e_k\}$. If we prove this we would have shown that G is eulerian. Suppose there is an edge $e = yz$ in G such that $e \neq e_i$ for any $i \in \{1, \ldots, k\}$. Then e is an edge in H. It follows that $\deg_H(y) \neq 0$ (also $\deg_H(z) \neq 0$). From (*) it follows that $y \notin \{x_1, x_2, \ldots, x_k\}$. Since G is connected, there is a path

$$\theta = (y_1, \ldots, y_r)$$

from y to x_1 ($y_1 = y$ and $y_r = x_1$). It follows that there is some $j \in \{2, 3, \ldots, r\}$ such that $y_j \in \{x_1, \ldots, x_k\}$ but $y_{j-1} \notin \{x_1, \ldots, x_k\}$ (j is the "first time" that the path θ hits the set $\{x_1, \ldots, x_k\}$). The edge $y_{j-1}y_j$ is in H since $y_{j-1} \notin \{x_1, \ldots, x_k\}$. This means $\deg_H(y_j) \geq 1$, which contradicts (*), since $y_j \in \{x_1, \ldots, x_k\}$. //

Hamilton graphs

Definition: Let $G = (V, E)$ be a graph. G is said to be hamiltonian if there exists a walk $\sigma = (x_1, x_2, \ldots, x_n)$

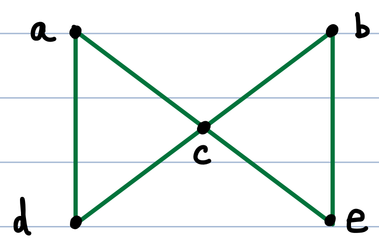
1. every vertex of G appears exactly once in σ
2. $x_n x_1 \in E$

Such a σ is called a hamiltonian cycle or a hamiltonian circuit.

Basic observations: Suppose $\sigma = (x_1, x_2, \ldots, x_n, x_{n+1})$ is a Hamilton circuit in a graph G. The following are easy to verify

1. If v is a vertex in G of degree 2, then both the edges incident on v must be part of σ, i.e. σ traverses both edges
2. σ is necessarily a cycle.
3. σ has no proper subcircuit, i.e. σ has no subcircuit which is not σ itself.
4. If v is a vertex in σ and e, f are edges in σ incident on v, then none of the other edges of G incident on v occurs in σ

Examples (non-existence of hamitonian cycle):

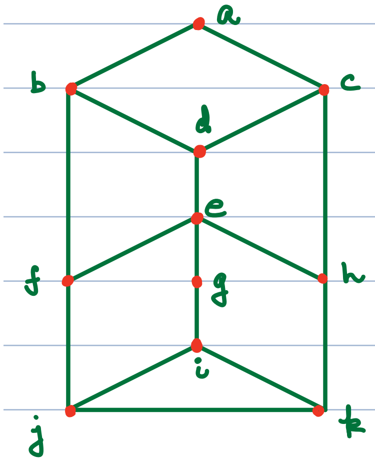
1. 

The graph displayed on the left does not have a Hamiltonian circuit as the following argument shows.

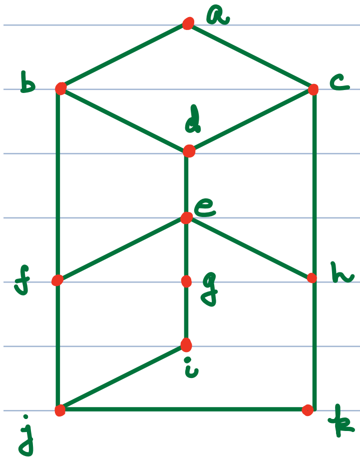
The degree 2 vertices are $a, b, d,$ and e.

If the graph has a hamiltonian cycle σ, by the first observation above, the edges ac, ad, bc, be, dc and ec are all part of σ. By the fourth observation only two edges incident on c can occur in σ. However, ac, bc, dc, and ec, all occur in σ. This is a contradiction. (Note that observations 2 and 3 are also violated, giving other proofs that a hamiltonian circuit does not exist.

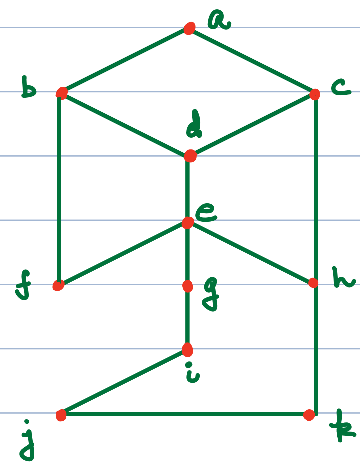
2. In the graph below suppose we had a hamiltonian circuit σ. There are only two vertices in the graph which



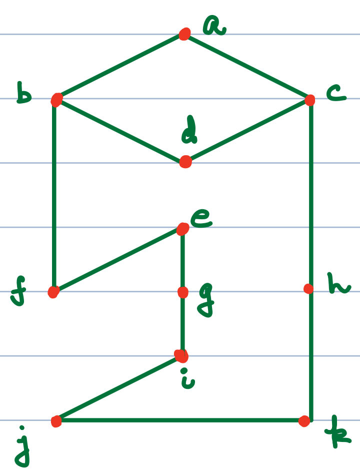
have degree 2, namely a and g. From our first observation, all edges incident on a and g are in σ. In particular eg and ig are both traversed by σ. Next, since ig is traversed by σ, by the fourth observation, exactly one of ij or ik is traversed by σ. Suppose ij is the edge traversed by σ. Then ik cannot be traversed by σ. Let us delete ik (see picture below). In the new graph, k has degree two and σ continues to be a hamiltonian circuit in the new graph. It follows that σ must traverse both jk and kh (by the first of our observations).



Next consider the vertex j. The edges ij and jk are traversed by σ. By our fourth observation, fj cannot be traversed by σ. Let us delete it. We now have the picture below. In this new picture, f has degree 2, and so σ must traverse bf and ef.

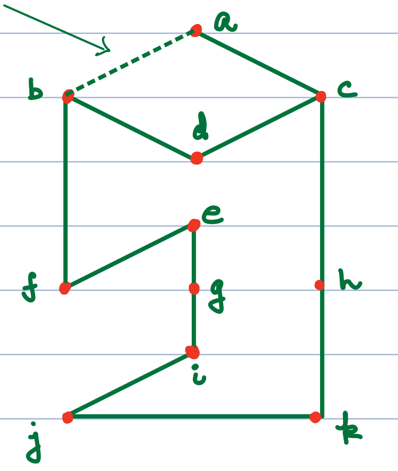


Now consider vertex e. We have seen that eg and fe lie on σ (i.e. σ traverses these two edges). This means that ed and eh are not traversed by σ, and so let us delete them (see picture below.



In the new graph d and h have degree 2, and so bd, cd, ch, and kh must lie on σ. Now bf and bd are traversed by σ, which means $ba = ab$ cannot be traversed by σ, by our our fourth observation

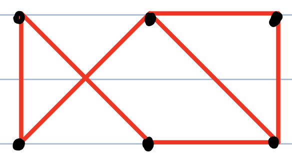
There and not there!



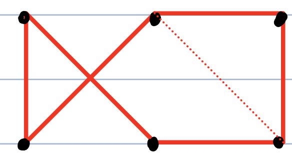
On the other hand $\deg(a) = 2$ and so by our first observation ab has to be traversed by σ. So we have a contradiction (the edge ab is on σ and it is not on σ at the same time). We arrived at this contradiction by assuming that the edge ij is on σ (and hence ik is not). By symmetry, if we had instead assumed ik was on σ and not ij, we would have again arrived at a contradiction. Thus no hamiltonian circuit exists on our graph.

Examples:

1.

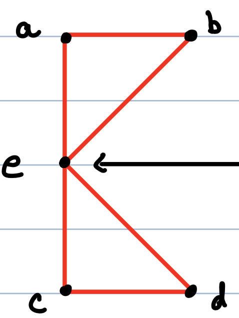


Hamitonian but not eulerian. Two vertices have degree 3.



Delete the edge given by the dotted line and you have a hamiltonian circuit.

2. Eulerian but not hamiltonian:



This vertex has to be visited at least twice by any circuit going through all vertices.

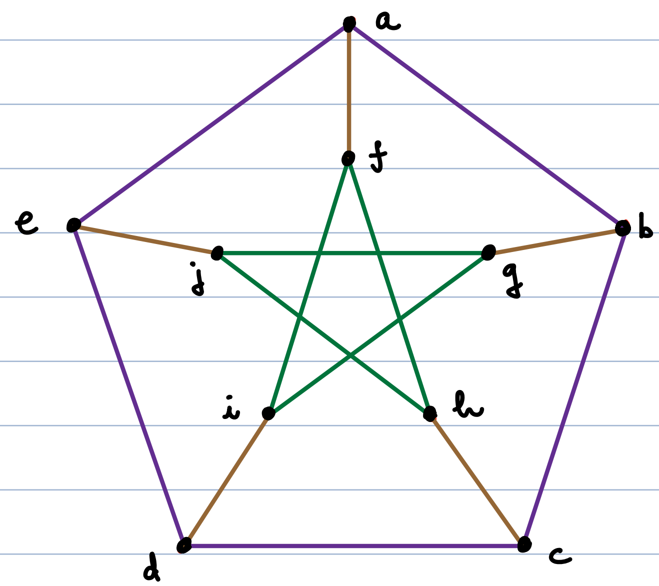
Suppose the graph above has a hamiltonian cycle. Call it σ. Since $\deg(a) = \deg(b) = \deg(c) = \deg(d) = 2$, the edges

ab, cd, ae, be, ce, and de

must all be traversed by σ. This means four edges in σ, namely ae, be, ce, and de, are incident to e. This violates the fourth observation above. Hence the graph is not hamiltonian.

3. The Petersen graph

The graph below is called the Petersen graph.



Some obvious properties:

1. It has 10 vertices and 15 edges.
2. Every vertex has degree 3.
3. It is connected
4. It has no cycles of length 3 or 4. All cycles are of length 5 or more.

From these observations it is not hard to see that the Petersen graph is not hamiltonian. An elementary proof involves a case by case elimination, and may be posted as a separate note later.