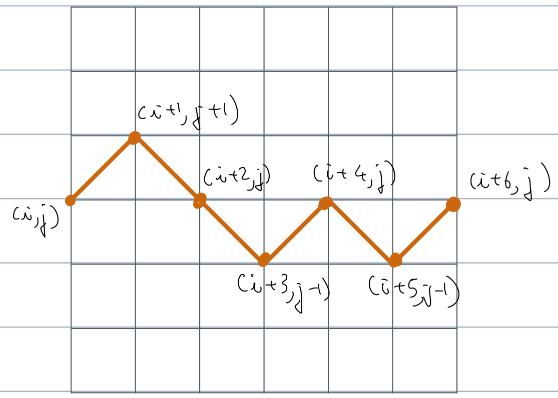
Supplementary note

Diagonal lattice paths in the plane

Definition : A **diagonal lattice path in the plane** is a curve made up of line segments that either go from a point (i,j) to the point $(i+1, j+1)$ or from a point (i,j) to the point $(i+1, j-1)$.



On the left is a typical diagonal lattice path. At each step (moving from left to right) the x-coordinate increases by one. The y-coordinate either increases by one or decreases by one.

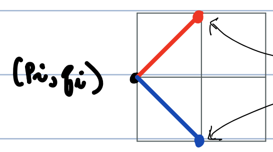
Another definition, equivalent to the one above is: A diagonal lattice path in the plane is a sequence of pairs of integers

$$(p_0, q_0), (p_1, q_1), \dots, (p_m, q_m)$$

such that for all $i = 0, 1, 2, \dots, k-1$, either

(a) $p_{i+1} = p_i + 1$ and $q_{i+1} = q_i + 1$ (a north east move)

(b) $p_{i+1} = p_i + 1$ and $q_{i+1} = q_i - 1$. (a south east move)



possibilities for (p_{i+1}, q_{i+1}). Red is the north east move and blue is the south east move.

Diagonal lattice paths and usual lattice paths are closely linked. There is a one-to-one correspondence between them. Suppose $(m_0, n_0), (m_1, n_1), \dots, (m_k, n_k)$ is a usual lattice path. Define (p_i, q_i) by the formula

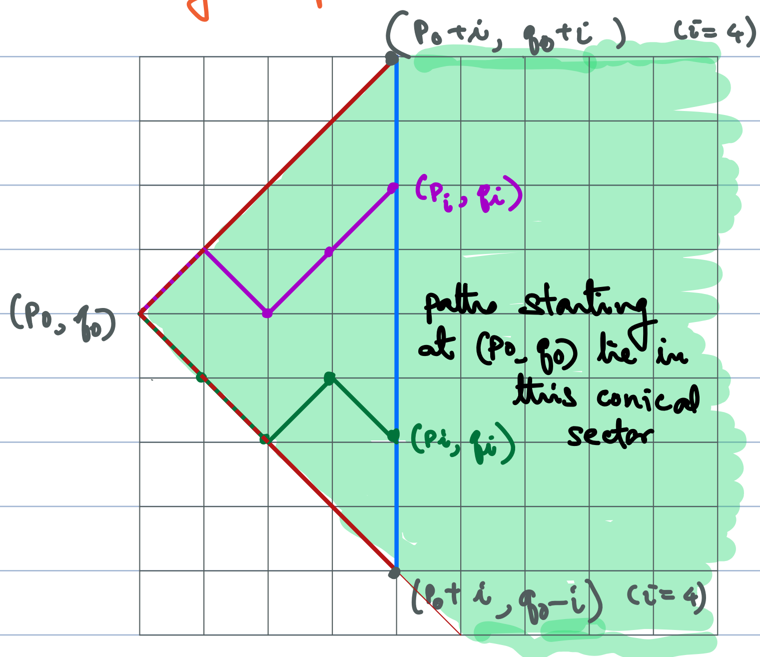
$$\begin{bmatrix} p_i \\ q_i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} m_i \\ n_i \end{bmatrix} \qquad i = 0, \dots, k.$$

Then $(p_0, q_0), \dots, (p_k, q_k)$ is a diagonal lattice path. Conversely, if $(p_0, q_0), \dots, (p_k, q_k)$ is a diagonal lattice path then $(m_0, n_0), \dots, (m_k, n_k)$ is a usual lattice paths where

$$\begin{bmatrix} m_i \\ n_i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} p_i \\ q_i \end{bmatrix}, \quad i = 0, \ldots, k.$$

Let $(p_0, q_0), \ldots, (p_m, q_m)$ be a diagonal lattice path. The following assertions are easy to verify and the proofs are left to you. If all else fails use the above formulas, though simple combinatorial thinking should do the trick.

1. $p_i = p_0 + i$, $0 \le i \le k$.
2. $p_0 - i \le q_i \le p_0 + i$, $0 \le i \le k$. The picture below may help.



The purple and the green paths are two possibilities. This is analogous to the fact that in usual lattice path $(m_0, n_0), \ldots, (m_k, n_k)$, $m_0 \le m_i \le m_0 + i$ and $n_0 \le n_i \le n_0 + i$.

3. Let the initial point (p_0, q_0) be the origin, and the terminal point (the last point) be (m, n), i.e.

$$(p_0, q_0) = (0, 0), \quad (p_m, q_m) = (m, n).$$

From 2 we know that $-m \le n \le m$ ← Important!

Let

u = number of north east moves

d = number of south east moves

Then

$$u + d = m, \quad u - d = n$$

and so

$$u = \frac{m+n}{2}, \quad d = \frac{m-n}{2}$$

4. Let $m \in \mathbb{N}_0$ and n an integer s.t. $-m \le n \le m$. Then the number of diagonal lattice paths from $(0,0)$ to (m,n) is

$$\binom{m}{\frac{m+n}{2}} = \binom{m}{\frac{m-n}{2}}.$$

5. Let $m \in \mathbb{N}_0$. If n does **not** satisfy the condition that $-m \le n \le m$, then there are no diagonal lattice paths from $(0,0)$ to (m,n). (This is obvious from 3. above).

6. If $p_0 + q_0$ is even, then all $p_i + q_i$ are even, and if $p_0 + q_0$ is odd, then all $p_i + q_i$ are odd.