MAT344 FALL 2022
PROBLEM SET 5

Due date: Dec 4, 2022 (on Crowdmark by midnight)

In what follows $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ denote the set of positive integers, the set of integers, and the set of real numbers respectively. $\mathbf{N}_0$ denotes the set of non-negative integers.

1. Let $n, m \in \mathbf{N}_0$ and $k \in \mathbf{N}$. Show using exponential generating functions that the number of ways of placing n distinct objects in $k+m$ distinct boxes $B_1, \ldots, B_k, B_{k+1}, \ldots, B_{k+m}$ so that there is at least one object in each of the boxes $B_1, \ldots, B_k$ is

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k+m-i)^n.$$

Note: Recall that one uses exponential generating functions when the order of the collection of objects being counted is important. See if you can translate the problem into one of $[k+m]$-strings of length n. There are other ways of doing this problem, but you are being tested on generating functions, and so you will get no credit unless you use exponential generating functions.

Solution: Fix $i \in \{1, \ldots, k\}$ and for $n \in \mathbf{N}_0$ let a_n be the number of ways one can distribute n distinct objects into the box B_i so that B_i has at least one object. When $n = 0$ there is clearly no way of distributing objects so that B_i has one object. So $a_0 = 0$. If $n \geq 1$, then there is only one way that these n distinct objects can be put into box B_i, namely we put all of them into B_i. Thus $a_n = 1$ for $n \geq 1$. It follows that the exponential generating function $E_i(x)$ for this $(a_n)_{n=0}^{\infty}$ is

$$E_i(x) = 0 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = e^x - 1, \quad i = 1, \ldots, k.$$

Next suppose $j \in \{k+1, \ldots, k+m\}$. Let b_n be the number of ways to distribute n distinct objects into box B_i. Since there are no restrictions, it is clear that $b_n = 1$ for all $n \geq 0$. If G_j is the exponential generating function for $(b_n)_{n=0}^{\infty}$ is is clear that

$$G_j(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = e^x, \quad j = k+1, \ldots, k+m.$$

Since the problem involves arrangements, therefore

$$\begin{aligned}
E(x) &= E_1(x)E_2(x)\dots E_k(x)G_{k+1}(x)G_{k+2}(x)\dots G_{k+m}(x) \\
&= (e^x-1)^k(e^x)^m \\
&= \left(\sum_{i=0}^{k}\binom{k}{i}(-1)^i e^{(k-i)x}\right)e^{mx} \\
&= \sum_{i=0}^{k}\binom{k}{i}(-1)^i e^{(m+k-i)x} \\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{k}\binom{k}{i}(-1)^i(m+k-i)^n\right)\frac{x^n}{n!}.
\end{aligned}$$

It follows that the number of ways to distribute n distinct objects to the $m+k$ boxes B_1, B_2, …, B_k, B_{k+1}, …, B_{k+m} so that B_1, …, B_k have at least one object is $\sum_{i=0}^{k}\binom{k}{i}(-1)^i(m+k-i)^n$, as asserted. □

Remark: Let $m=0$. Check that you recover the formula for the number of surjective maps from $[n]$ to $[k]$ (see Lecture 14). You can regard a distribution of the n distinct objects $p_1,\dots,p_n$ into the given k boxes as a map$f$ from $[n]$ to $[k]$ with $f(r)=i$ if p_r is placed in B_i.

2. Find the number of $\{A,B,C,D\}$-strings of length $n\in\mathbf{N}$ containing an odd number of As, an even number of Bs, and at least one C.

Solution: Let $n\in\mathbf{N}_0$.

Let a_n be the number of strings of length n consisting only A's and such that there are odd number of A's in the string. Clearly $a_n=0$ when n is even and $a_n=1$ when n is odd. Let E_A be the exponential generating function for $(a_n)_{n=0}^{\infty}$. Then

$$E_A(x)=\frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\dots=\frac{e^x-e^{-x}}{2}.$$

Let b_n be the number of strings of length n consisting of only B's such that the number of B's in the string is even. It is clear that $b_n=1$ when n is even and $b_n=0$ when n is odd. Let E_B be the exponential generating function for $(b_n)_{n=0}^{\infty}$. Then

$$E_B(x)=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots=\frac{e^x+e^{-x}}{2}.$$

Let c_n be the number of strings of length n consisting only of C's, such that there is at least one C in the string, and let E_C be the exponential generating function for $(c_n)_{n=0}^{\infty}$. Clearly $c_0=0$ and $c_n=1$ for $n\geq 1$. Thus

$$E_C=0+\frac{x}{1!}+\frac{x^2}{2!}+\dots=e^x-1.$$

If d_n is the number of strings of length n consisting of only D's, then $d_n=1$ for all $n\in\mathbf{N}_0$, and the corresponding exponential generating function for the sequence $(d_n)_{n=0}^{\infty}$ is

$$E_D(x)=e^x.$$

Let p_n be the number of $\{A, B, C, D\}$-strings of length n which have an odd number of A's, an even number of B's, and at least one C, and let $E(x)$ be the exponential generating function of $(p_n)_{n=0}^{\infty}$. Then

$$\begin{aligned}
E(x) &= E_A(x)E_B(x)E_C(x)E_D(x) \\
&= \frac{e^x - e^{-x}}{2}\frac{e^x + e^{-x}}{2}(e^x - 1)e^x \\
&= \frac{e^{4x} - e^{3x} + e^{-x}}{4} - \frac{1}{4} \\
&= \sum_{n=0}^{\infty} \frac{4^n - 3^n + (-1)^n}{4}\frac{x^n}{n!} - \frac{1}{4} \\
&= \frac{4^0 - 3^0 + (-1)^0}{4} - \frac{1}{4} + \sum_{n=1}^{\infty} \frac{4^n - 3^n + (-1)^n}{4}\frac{x^n}{n!} \\
&= \sum_{n=1}^{\infty} \frac{4^n - 3^n + (-1)^n}{4}\frac{x^n}{n!}.
\end{aligned}$$

Thus,

$$p_n = \begin{cases} 0 & \text{if } n = 0 \\ \dfrac{4^n - 3^n + (-1)^n}{4} & \text{if } n \geq 1. \end{cases}$$

□

3. For all $n \in \mathbf{N}$, let a_n be the number of ternary strings of length n where no two consecutive digits are non-zero. For example, if $n = 8$, then 01000201 and 10201020 are valid, but 01002100 is not valid since 21 are two consecutive digits which are non-zero. Also, 10200110 is not valid because of 11.

(a) Find a recurrence relation for a_n.

(b) Solve for a_n using the method of advancement operators.

Note: Since $n \in \mathbf{N}$, there is no a_0. You need to find a_1 and a_2 for your initial conditions.

Solution: For $n \in \mathbf{N}$, let S_n be the set of ternary strings of length n which do not have two consecutive non-zero digits.

Let $n \geq 3$. If a string $x_1x_2 \ldots x_n \in S_n$ starts with 0 (i.e. if $x_1 = 0$), then $x_2 \ldots x_n$ is in S_{n-1}. If on the other hand $x_1 \neq 0$, then necessarily $x_2 = 0$ and $x_3 \ldots x_n \in S_{n-2}$. There are two choices for x_1. It follows from the above observations that $|S_n| = |S_{n-1}| + 2|S_{n-2}|$. Thus

$$a_n = a_{n-1} + 2a_{n-2}, \qquad n \geq 3.$$

Moreover, $S_1 = \{0, 1, 2\}$ and $S_2 = \{00, 01, 02, 10, 20\}$, whence $a_1 = 3$ and $a_2 = 5$. The characteristic polynomial for our recurrence relation is $x^2 - x - 2 = (x-2)(x+1)$ and hence the general solution is $a_n = c(-1)^n + d2^n$ for some constants c and d. Since $a_1 = 3$, and $a_2 = 5$, we get

$$-c + 2d = 3, \quad \text{and} \quad c + 4d = 5.$$

Thus $c = -1/3$ and $d = 4/3$. Thus

$$a_n = -\frac{1}{3}(-1)^n + \frac{4}{3}2^n, \quad n \in \mathbf{N}.$$

This can be simplified (if you wish) to

$$a_n = \frac{(-1)^{n+1} + 2^{n+2}}{3}, \quad n \in \mathbf{N}.$$

You will not lose any marks for not simplifying. □

4. Let $(a_n)_{n=0}^{\infty}$ be the sequence defined by

$$a_n = 3a_{n-1} + 2a_{n-2}, \quad \text{for all } n \geq 2,$$

with initial conditions, $a_0 = 0$, and $a_1 = 1$. Use the method of generating functions to find a_n, $n \in \mathbf{N}_0$.

Solution: The recurrence relation gives the equality of power series

$$\sum_{n=2}^{\infty} a_n x^n = 3x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + 2x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2}.$$

Let $g(x)$ be the generating function for $(a_n)_{n=0}^{\infty}$. Since $a_0 = 0$ and $a_1 = 1$, we have $\sum_{n=2}^{\infty} a_n x^n = g(x) - a_0 - a_1 x = g(x) - x$. Similarly $\sum_{n=2}^{\infty} a_{n-1} x^{n-1} = g(x) - a_0 = g(x)$, and $\sum_{n=2}^{\infty} a_{n-2} x^{n-2} = g(x)$. Substituting in the above, we get

$$(1 - 3x - 2x^2)g(x) = x,$$

i.e.

$$g(x) = \frac{x}{1 - 3x - 2x^2}.$$

As mentioned in class, the roots of $1 - 3x - 2x^2$ are the reciprocals of the roots of $x^2 - 3x - 2$, for if r is a root of $x^2 - 3x - 2$, we have $r^2 - 3r - 2 = 0$, which gives, on multiplying by r^{-2}, $1 - 3r^{-1} - 2r^{-2} = 0$. The roots of $x^2 - 3x - 2$ are

$$\alpha = \tfrac{1}{2}(3 + \sqrt{17}), \quad \text{and} \quad \beta = \tfrac{1}{2}(3 - \sqrt{17}).$$

From this it follows (since α^{-1} and β^{-1} are the roots of $1 - 3x - 2x^2$) that $1 - 3x - 2x^2 = (\alpha x - 1)(\beta x - 1)$. We have the partial fraction decomposition

$$\frac{x}{(\alpha x - 1)(\beta x - 1)} = \frac{1}{\alpha - \beta}\left\{\frac{1}{1 - \alpha x} - \frac{1}{1 - \beta x}\right\}.$$

Thus

$$\begin{aligned} g(x) &= \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} (\alpha^n - \beta^n) x^n \\ &= \frac{1}{\sqrt{17}} \sum_{n=0}^{\infty} \left\{ \frac{(3 + \sqrt{17})^n - (3 - \sqrt{17})^n}{2^n} \right\} x^n. \end{aligned}$$

The formula for a_n is therefore:

$$a_n = \frac{(3 + \sqrt{17})^n - (3 - \sqrt{17})^n}{2^n \sqrt{17}}, \quad n \in \mathbf{N}_0.$$

□

5. In a bridge hand, you and your partner (whose hand you know) have seven clubs between the two of you. What is the probability that one of your opponents has exactly four of the remaining clubs?

Solution: Let us first calculate the probability that the opponent on your left has exactly four of the remaining clubs.

Of the 26 cards your opponents have between them, 20 of them are non-clubs. The opponent on the left gets $13-4=9$ of them. There are $\binom{20}{9}$ ways of distributing nine cards out the 20 to this opponent. To these nine cards we have to add four clubs chosen from the six clubs available to your opponents. There are $\binom{6}{4}$ ways of distributing four clubs to the opponent on your left. Thus there are $\binom{6}{4}\binom{20}{9}$ ways of distributing 13 cards to the opponent on your left so that he or she has exactly four of the clubs available to your opponents. Therefore, assuming all $\binom{26}{13}$ possible hands are equally possible for this opponent, the probability that he or she has exactly four of the remaining clubs is $\frac{\binom{6}{4}\binom{20}{9}}{\binom{26}{13}}$.

By symmetry, the probability that your opponent on your right has exactly four of the six clubs available to your opponents is also $\frac{\binom{6}{4}\binom{20}{9}}{\binom{26}{13}}$.

The two events are mutually exclusive, since it is not possible that both your opponents have exactly four clubs. Hence the required probability is

$$2\frac{\binom{6}{4}\binom{20}{9}}{\binom{26}{13}}.$$

□

6. A process sends an object along **diagonal lattice paths** on the xy-plane. This means the object only moves in the northeast or southeast directions, and can only change direction at lattice points, i.e. points with integer coordinates. Suppose the process is random and sends the object from $A=(0,\,0)$ to $B=(12,\,2)$ in such a way that every diagonal lattice path from A to B has an equal chance of being travelled on by the object. What is the probability that the object will travel along the segment from (5, 3) to (6, 2)?

Solution: Let $P=(5,\,3)$ and $Q=(6,\,2)$. Any diagonal lattice path from A to B which passes through the segment from P to Q can be broken up into three diagonal lattice paths: One from A to P, one from P to Q, and one from Q to B.

There is only one diagonal lattice path from P to Q, by definition of a diagonal lattice path. The number of diagonal lattice paths from A to P is $\binom{5}{4}$. The number of diagonal lattice paths from Q to B is the same as the number of paths from (0, 0) to (6, 0) and this number is $\binom{6}{3}$.

Thus the number of diagonal lattice paths from A to B which have PQ as a segment is $\binom{5}{4}\binom{6}{3}$. The total number of diagonal lattice paths from A to B is $\binom{12}{7}$. So the answer is

$$\frac{\binom{5}{4}\binom{6}{3}}{\binom{12}{7}} \approx 0.126.$$

It is enough to leave the answer as the expression in binomial coefficients. □