MAT344 FALL 2022
PROBLEM SET 5

Due date: Dec 4, 2022 (on Crowdmark by midnight)

In what follows $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ denote the set of positive integers, the set of integers, and the set of real numbers respectively. $\mathbf{N}_0$ denotes the set of non-negative integers.

1. Let $n, m \in \mathbf{N}_0$ and $k \in \mathbf{N}$. Show using exponential generating functions that the number of ways of placing n distinct objects in $k+m$ distinct boxes $B_1, \ldots, B_k, B_{k+1}, \ldots, B_{k+m}$ so that there is at least one object in each of the boxes $B_1, \ldots, B_k$ is

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k+m-i)^n.$$

Note: Recall that one uses exponential generating functions when the order of the collection of objects being counted is important. See if you can translate the problem into one of $[k+m]$-strings of length n. There are other ways of doing this problem, but you are being tested on generating functions, and so you will get no credit unless you use exponential generating functions.

2. Find the number of $\{A, B, C, D\}$-strings of length $n \in \mathbf{N}$ containing an odd number of As, an even number of Bs, and at least one C.

3. For all $n \in \mathbf{N}$, let a_n be the number of ternary strings of length n where no two consecutive digits are non-zero. For example, if $n = 8$, then 01000201 and 10201020 are valid, but 01002100 is not valid since 21 are two consecutive digits which are non-zero. Also, 10200110 is not valid because of 11.

(a) Find a recurrence relation for a_n.

(b) Solve for a_n using the method of advancement operators.

Note: Since $n \in \mathbf{N}$, there is no a_0. You need to find a_1 and a_2 for your initial conditions.

4. Let $(a_n)_{n=0}^{\infty}$ be the sequence defined by

$$a_n = 3a_{n-1} + 2a_{n-2}, \quad \text{for all } n \geq 2,$$

with initial conditions, $a_0 = 0$, and $a_1 = 1$. Use the method of generating functions to find a_n, $n \in \mathbf{N}_0$.

5. In a bridge hand, you and your partner (whose hand you know) have seven clubs between the two of you. What is the probability that one of your opponents has exactly four of the remaining clubs?

6. A process sends an object along **diagonal lattice paths** on the xy-plane. This means the object only moves in the northeast or southeast directions, and can only change direction at lattice points, i.e. points with integer coordinates. Suppose the process is random and sends the object from $A = (0, 0)$ to $B = (12, 2)$ in such a way that every diagonal lattice path from A to B has an equal chance of being travelled on by the object. What is the probability that the object will travel along the segment from $(5, 3)$ to $(6, 2)$?