MAT344 FALL 2022
PROBLEM SET 4

Due date: Nov 20, 2022 (on Crowdmark by midnight)

In what follows $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ denote the set of positive integers, the set of integers, and the set of real numbers respectively. $\mathbf{N}_0$ denotes the set of non-negative integers.

1. Suppose $(a_n)_{n=0}^{\infty}$ satisfies the recursion relation $a_n = (n-1)(a_{n-1}+a_{n-2})$, $n \geq 2$, with initial conditions: $a_0 = 1$ and $a_1 = 0$. Show that $(a_n)_{n=0}^{\infty}$ also satisfies the recurrence relation

$$a_n = na_{n-1} + (-1)^n, \qquad n \in \mathbf{N}.$$

Hint: Consider $b_n = a_n - na_{n-1}$, $n \in \mathbf{N}$, $b_0 = 1$. Find a suitable recurrence relation for $(b_n)_{n=0}^{\infty}$ and solve that recurrence relation.

Solution: Let b_n, $n \in \mathbf{N}_0$, be as in the hint. Since $a_0 = 1$ and $a_1 = 0$, we have $b_1 = 0 - (1)(1) = -1$. Thus $b_1 = -b_0$. Now suppose $n \geq 2$. Then

$$\begin{aligned} b_n &= a_n - na_{n-1} \\ &= \big(a_n - (n-1)a_{n-1}\big) - a_{n-1} \\ &= (n-1)a_{n-2} - a_{n-1} \\ &= -b_{n-1} \end{aligned}$$

Thus $b_n = -b_{n-1}$ for $n \geq 1$ (we checked this for $n = 1$ earlier), and the initial condition is $b_0 = 1$. It then follows that $b_n = (-1)^n$, $n \in \mathbf{N}_0$.[1] Since $a_n - na_{n-1} = b_n$ for $n \in \mathbf{N}$, it follows that $a_n - na_{n-1} = (-1)^n$ for $n \in \mathbf{N}$. This proves the assertion in the question. □

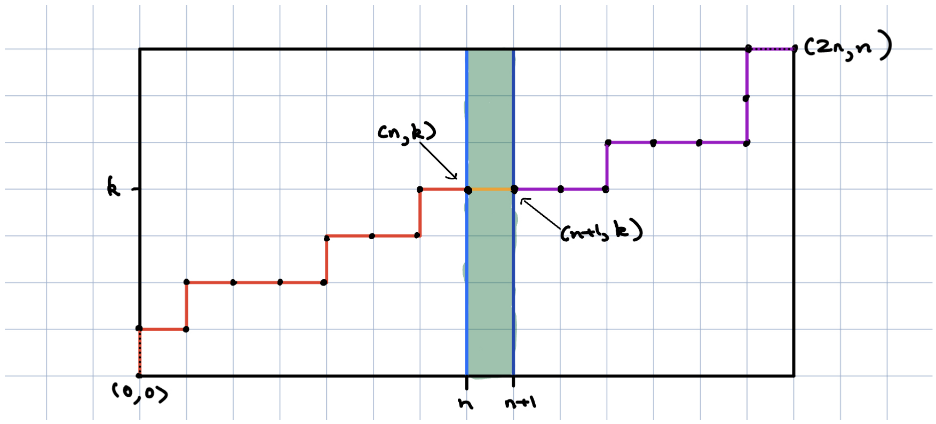
2. Give a lattice path proof of

$$\binom{3n}{2n} = \sum_{k=0}^{n} \binom{n+k}{n}\binom{2n-k-1}{n-1}, \qquad n \in \mathbf{N}.$$

Hint: Consider (usual, not diagonal) lattice paths from $(0,0)$ to (a,b) for a suitable lattice point (a,b), such that the total number of such paths is the left side. Count these paths in a different way by looking at the first time such a path hits a suitable vertical line $x = c$ (or, depending on your choice of (a,b), a suitable horizontal line $y = c$). With the right choice of a, b, and c, you should be able to do the problem. Something similar (but with diagonal lattice paths rather than usual lattice paths) was suggested in the "*Problems Worth Thinking About*" section of the plans for week 8.

[1] One doesn't need any advanced theory for this. Just use the fact that the initial value of b_n is 1, and successive values are negatives of each other.

Solution: Let S be the set of lattice paths from $(0, 0)$ to $(2n,\, n)$. Then $|S| = \binom{3n}{2n}$. For $k \in \{0, 1, \ldots, n\}$ let S_k be the subset of S consisting of those paths in S which hit the line $x = n + 1$ for the first time at $(n + 1,\, k)$. Equivalently, S_k is the set of paths which cross the region between the vertical lines $x = n$ and $x = n + 1$ at "level k", i.e. along the line segment from $(n,\, k)$ to $(n + 1,\, k)$. The following picture (with $n = 7$ and $k = 4$) may help. The point is that to get from $(0, 0)$ to $(2n, n)$, you have to cross the green stream, and you can only cross the stream along bridges like the orange one.



It is clear that every path in S lies in a unique subset S_k, and hence $S = \bigcup_{k=0}^{n} S_k$, with $S_i \cap S_j = \emptyset$ if $i \neq j$. Thus

$$|S| = \sum_{k=0}^{n} |S_k|.$$

Now an element in S_k is the same as a lattice path from $(0, 0)$ to (n, k), followed by the line segment from (n, k) to $(n + 1, k)$, followed by a lattice path from $(n + 1, k)$ to $(2n,\, n)$. In other words it is completely determined by the red and purple paths in the picture (since the orange line segment is a must for all paths in S_k). The number of red paths is $\binom{n+k}{n}$ and the number of purple paths is the same as the number of lattice paths from $(0,\, 0)$ to $(2n - (n + 1),\, n - k) = (n - 1, n - k)$. This number is $\binom{2n-k-1}{n-1}$. So $|S_k| = \binom{n+k}{n}\binom{2n-k-1}{n-1}$. Thus

$$\binom{3n}{2n} = |S| = \sum_{k=0}^{n} |S_k| = \sum_{k=0}^{n} \binom{n+k}{n}\binom{2n-k-1}{n-1}.$$

as required.

Note: One can also look at lattice paths from $(0,\, 0)$ to $(n,\, 2n)$. Then our green stream will be horizontal. □

3. Let $G = (V,\, E)$ be a graph such that $2 \leq |V| < \infty$. Suppose $2 \deg_G(v) \geq |V| - 1$ for all $v \in V$. Show that G is connected. **Hint:** Try a proof by contradiction.

Solution: Suppose G is not connected. Then there exist $x, y \in V$, $x \neq y$, such that there is no path from x to y. For any vertex v of G, let N_v denote the neighbourhood of v, i.e. N_v is the set of vertices in G adjacent to v. Note that $\deg_G(v) = |N_v|$. Since there is no path from x to y, it is clear that $N_x \cap N_y = \emptyset$. For the same reason, $y \notin N_x$ and $x \notin N_y$. Thus N_x, N_y, and $\{x, y\}$ are three disjoint subsets of V. It follows that

$$|V| \geq |N_x| + |N_y| + |\{x, y\}| \geq \tfrac{1}{2}(|V| - 1) + \tfrac{1}{2}(|V| - 1) + 2 = |V| + 1$$

which is impossible. Hence G is connected. □

4. Let X be a finite set and $A_1, A_2, \ldots, A_n$ subsets of X. For any subset T of $[n]$, let $A_T = \bigcap_{j \in T} A_j$, with the understanding that if $T = \emptyset$, then $A_T = X$. Fix a subset S of $[n]$. Let Y be the subset of X consisting of all elements of X which belong to A_i for every $i \in S$, but for no other indices. In other words,

$$Y = \{x \in X \mid x \in A_S \text{ and } x \notin A_i \text{ if } i \notin S\}.$$

Show that

$$|Y| = \sum_{S \subset T \subset [n]} (-1)^{|T \setminus S|} |A_T|.$$

The sum is taken over subsets T of $[n]$ which contain S.

Solution: Let $D = [n] \setminus S$. For $i \in D$, let $B_i = A_i \cap A_S$. Then

$$Y = A_S \setminus \left(\bigcup_{i \in D} B_i \right).$$

By the Inclusion-Exclusion principle

$$|Y| = \sum_{E \subset D} (-1)^{|E|} \left| \bigcap_{i \in E} B_i \right|,$$

with the understanding that if E is empty then $\bigcap_{i \in E} B_i = A_S$. There is a bijective correspondence between subsets of D and subsets of $[n]$ which contain S, namely a subset E of D corresponds to $T = S \cup E$, and a subset T of $[n]$ which contains S corresponds to $E = T \setminus S$. It is straightforward to check the process of going from E to T is the inverse of the process of going from T to E. Moreover, for E a subset of D,

$$\bigcap_{i \in E} B_i = \bigcap_{i \in E} (A_i \cap A_S) = A_{S \cup E}.$$

From these observations, the formula for $|Y|$ above translates to the formula below:

$$|Y| = \sum_{S \subset T \subset [n]} (-1)^{|T \setminus S|} |A_T|.$$

□

5. For $n \in \mathbf{N}$, let Δ_n be the number of derangements of $[n]$ and set $\Delta_0 = 1$. Recall that we proved in class that

$$\Delta_n = \sum_{k=0}^{n} (-1)^n \binom{n}{k} (n-k)! = n! \sum_{k=0}^{n} (-1)^k / k!.$$

Show combinatorially, *without using the above formulas*, that

$$\Delta_n = (n-1)(\Delta_{n-1} + \Delta_{n-2})$$

for $n \geq 2$.

Solution: For clarity, the solution is more elaborate than usual.

Case $n = 2$**:** It is clear that there are no derangements possible for the set [1]. This means $\Delta_1 = 0$. Also the only derangement possible for $[2] = \{1, 2\}$ is the permutation which sends 1 to 2 and 2 to 1. Thus $\Delta_2 = 1$. It is immediate that $\Delta_2 = (2-1)(\Delta_1 + \Delta_0)$, since, by definition, $\Delta_0 = 1$. We are therefore done for the case $n = 2$.

Case $n \geq 3$**:** Suppose $n \geq 3$. Let D_n be the set of derangements of $[n]$ and for $i = 2, 3, \ldots, n$. let $S_i = \{f \in D_n \mid f(1) = i\}$. Then the S_i are pairwise disjoint, i.e. $S_i \cap S_i = \emptyset$ for $i \neq j$, $i, j \in \{2, 3, \ldots, n\}$. Moreover, $D_n = \bigcup_{i=2}^n S_i$.

For each $i \in \{2, 3, \ldots, n\}$, S_i breaks up further into disjoint subsets A_i and B_i, where

$$A_i = \{f \in S_i \mid f(i) = 1\}$$

and

$$B_i = \{f \in S_i \mid f(i) \neq 1\}.$$

Claim: $|A_i| = \Delta_{n-2}$ for $i = 2, 3, \ldots, n$.

Proof. The restriction of any $f \in A_i$ to $[n] \smallsetminus \{1, i\}$ is a derangement of $[n] \smallsetminus \{1, i\}$, and conversely, given a derangement g of $[n] \smallsetminus \{1, i\}$, the map $f \colon [n] \to [n]$ given by $f(1) = i$, $f(i) = 1$, and $f(x) = g(x)$ for $x \in [n] \smallsetminus \{1, i\}$, is a derangement of $[n]$ which is in A_i. Thus

$$|A_i| = \Delta_{n-2}, \quad i = 2, 3, \ldots, n,$$

as claimed.

Claim: $|B_i| = \Delta_{n-1}$, $i = 2, 3, \ldots, n$.

Proof. Suppose $f \in B_i$. Let $j = f^{-1}(1)$. Then j is not i, by definition of B_i, and j is not 1 since f is a derangement. Define $g \colon [n] \smallsetminus \{1\} \to [n] \smallsetminus \{1\}$ by the rule

$$g(x) = \begin{cases} f(x) & \text{when } x \neq j; \\ i & \text{when } x = j. \end{cases}$$

Then g is a derangement of $[n] \smallsetminus \{1\}$. Conversely, given a derangement g of $[n] \smallsetminus \{1\}$, with $j = g^{-1}(i)$, we can define an element f of B_i by setting $f(1) = i$, $f(j) = 1$, and $f(x) = g(x)$ for $x \in [n] \smallsetminus \{1, j\}$. Thus the correspondence $f \mapsto g$ described above gives a bijective correspondence between B_i and the set of derangements of $[n] \smallsetminus \{1\}$. It follows that

$$|B_i| = \Delta_{n-1}, \quad i = 2, 3, \ldots, n,$$

proving the claim.

Since $|S_i| = |A_i| + |B_i|$, we therefore have $|S_i| = \Delta_{n-2} + \Delta_{n-1}$. In other words,

$$\begin{aligned} \Delta_n = |D_n| = \sum_{i=2}^n |S_i| &= \sum_{i=2}^n (\Delta_{n-1} + \Delta_{n-2}) \\ &= (\Delta_{n-1} + \Delta_{n-2}) \sum_{n=2}^n 1 \\ &= (\Delta_{n-1} + \Delta_{n-2})(n-1), \end{aligned}$$

proving the assertion of the problem. (We have used the fact that $n \geq 3$ by finding distinct elements 1, i, and j in the above argument.) $\square$

6. For $n \in \mathbf{N}_0$, let a_n be the number of non-negative integer solutions of

$$3a + b + 7c + d = n$$

with the added conditions that $b \leq 2$, $c \geq 1$, and $d \leq 6$. Find the generating function for $(a_n)_{n=0}^{\infty}$. Use this to determine a_n for $n \in \mathbf{N}_0$.

Solution: For $n \in \mathbf{N}_0$, let α_n be the number of non-negative integer solutions of $3a = n$, β_n the number of integer solutions of $b = n$ with $0 \leq b \leq 2$, γ_n the number of integer solutions of $7c = n$ with $c \geq 1$, and δ_n the number of integer solutions of $d = n$ with $d \geq 6$. It is clear that $\alpha_n = 0$ if n is not a multiple of 3 and $\alpha_{3k} = 1$ for $k \in \mathbf{N}_0$; $\beta_n = 1$ for $0 \leq n \leq 2$ and $\beta_n = 0$ for $n \geq 3$; $\gamma_0 = 0$, $\gamma_n = 0$ if n is not a multiple of 7, and $\gamma_{7k} = 1$ for $k \geq 1$; and finally $\delta_n = 1$ for $n \leq 6$ and $\delta_n = 0$ for $n \geq 7$. Let F_α, F_β, F_γ, and F_δ be the generating functions of $(\alpha_n)_{n=0}^{\infty}$, $(\beta_n)_{n=0}^{\infty}$, $(\gamma_n)_{n=0}^{\infty}$, and $(\delta_n)_{n=0}^{\infty}$ respectively, then

$$F_\alpha(x) = \sum_{n=0}^{\infty} \alpha_n x^n = 1 + x^3 + x^6 + \cdots = \sum_{n=0}^{\infty} (x^3)^n = \frac{1}{1 - x^3}$$

$$F_\beta(x) = \sum_{n=0}^{\infty} \beta_n x^n = 1 + x + x^2 = \frac{1 - x^3}{1 - x}.$$

$$F_\gamma(x) = \sum_{n=0}^{\infty} \gamma_n x^n = x^7 + x^{14} + x^{21} + \cdots = \sum_{n=1}^{\infty} (x^7)^n = x^7 \sum_{n=0}^{\infty} (x^7)^n = \frac{x^7}{1 - x^7}.$$

$$F_\delta(x) = \sum_{n=0}^{\infty} \delta_n x^n = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 = \frac{1 - x^7}{1 - x}.$$

Let $(a_n)_{n=0}^{\infty}$ be as in the problem and $F(x)$ its generating function. Then

$$\begin{aligned} F(x) &= F_\alpha(x) F_\beta(x) F_\gamma(x) F_\delta(x) \\ &= \frac{1}{1 - x^3} \frac{1 - x^3}{1 - x} \frac{x^7}{1 - x^7} \frac{1 - x^7}{1 - x} \\ &= \frac{x^7}{(1 - x)^2} \\ &= x^7 \sum_{m=0}^{\infty} (m + 1) x^m \\ &= \sum_{n=7}^{\infty} (n - 6) x^n. \end{aligned}$$

Thus

$$a_n = \begin{cases} 0 & \text{when } n \leq 6 \\ n - 6 & \text{when } n \geq 7. \end{cases}$$

$\square$