MAT344 FALL 2022
PROBLEM SET 4

Due date: Nov 20, 2022 (on Crowdmark by midnight)

In what follows $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ denote the set of positive integers, the set of integers, and the set of real numbers respectively. $\mathbf{N}_0$ denotes the set of non-negative integrs.

1. Suppose $(a_n)_{n=0}^{\infty}$ satisfies the recursion relation $a_n = (n-1)(a_{n-1}+a_{n-2})$, $n \geq 2$, with initial conditions: $a_0 = 1$ and $a_1 = 0$. Show that $(a_n)_{n=0}^{\infty}$ also satisfies the recurrence relation

$$a_n = na_{n-1} + (-1)^n, \qquad n \in \mathbf{N}.$$

Hint: Consider $b_n = a_n - na_{n-1}$, $n \in \mathbf{N}$, $b_0 = 1$. Find a suitable recurrence relation for $(b_n)_{n=0}^{\infty}$ and solve that recurrence relation.

2. Give a lattice path proof of

$$\binom{3n}{2n} = \sum_{k=0}^{n} \binom{n+k}{n}\binom{2n-k-1}{n-1}, \qquad n \in \mathbf{N}.$$

Hint: Consider (usual, not diagonal) lattice paths from $(0,0)$ to (a,b) for a suitable lattice point (a,b), such that the total number of such paths is the left side. Count these paths in a different way by looking at the first time such a path hits a suitable vertical line $x = c$ (or, depending on your choice of (a,b), a suitable horizontal line $y = c$). With the right choice of a, b, and c, you should be able to do the problem. Something similar (but with diagonal lattice paths rather than usual lattice paths) was suggested in the "*Problems Worth Thinking About*" section of the plans for week 8.

3. Let $G = (V, E)$ be a graph such that $2 \leq |V| < \infty$. Suppose $2\deg_G(v) \geq |V|-1$ for all $v \in V$. Show that G is connected. **Hint:** Try a proof by contradiction.

4. Let X be a finite set and $A_1, A_2, \ldots, A_n$ subsets of X. For any subset T of $[n]$, let $A_T = \bigcap_{j\in T} A_j$, with the understanding that if $T = \emptyset$, then $A_T = X$. Fix a subset S of $[n]$. Let Y be the subset of X consisting of all elements of X which belong to A_i for every $i \in S$, but for no other indices. In other words,

$$Y = \{x \in X \mid x \in A_S \text{ and } x \notin A_i \text{ if } i \notin S\}.$$

Show that

$$|Y| = \sum_{S\subset T\subset[n]} (-1)^{|T\setminus S|}|A_T|.$$

The sum is taken over subsets T of $[n]$ which contain S.

5. For $n \in \mathbf{N}$, let Δ_n be the number of derangements of $[n]$ and set $\Delta_0 = 1$. Recall that we proved in class that

$$\Delta_n = \sum_{k=0}^{n} (-1)^n \binom{n}{k} (n-k)! = n! \sum_{k=0}^{n} (-1)^k / k!.$$

Show combinatorially, *without using the above formulas*, that

$$\Delta_n = (n-1)(\Delta_{n-1} + \Delta_{n-2})$$

for $n \geq 2$.

Note that the recurrence relation for (Δ_n) is the same as that given at the beginning of Problem **1**. This fact will be useful later in the course.

6. For $n \in \mathbf{N}_0$, let a_n be the number of non-negative integer solutions of

$$3a + b + 7c + d = n$$

with the added conditions that $b \leq 2$, $c \geq 1$, and $d \leq 6$. Find the generating function for $(a_n)_{n=0}^{\infty}$. Use this to determine a_n for $n \in \mathbf{N}_0$.