MAT344 FALL 2022
PROBLEM SET 3

Due date: Nov 6, 2022 (on Crowdmark by midnight)

In what follows **N**, **Z**, **R** denote the set of positive integers, the set of integers, and the set of real numbers respectively. $\mathbf{N}_0$ denotes the set of non-negative integrs.

Bipartite graphs. A graph $G = (V, E)$ is said to be *bipartite* if there are two non-empty subsets V_1 and V_2 such that $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = V$ and every edge joins a vertex in V_1 to a vertex in V_2. It is easy to see that G is bipartite if and only if every walk of the form $(x_1, \ldots, x_n)$ with $x_n = x_1$ has even length. You don't have to submit a proof of this (easy) fact, but you might try and work out a proof for yourself. One typically tries to draw the vertices of V_1 on the left and V_2 on the right as below:

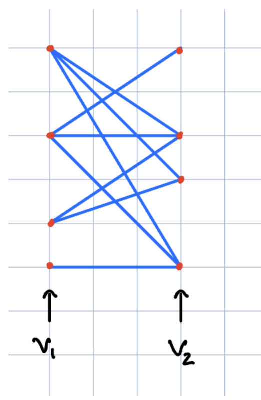


FIGURE 1. A bipartite graph drawn with V_1 on the left and V_2 on the right

Sometimes it is easier (and uses less paper) to draw it as in FIGURE 2.

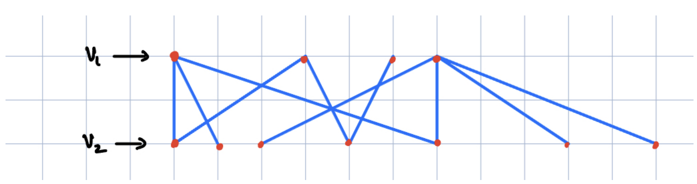
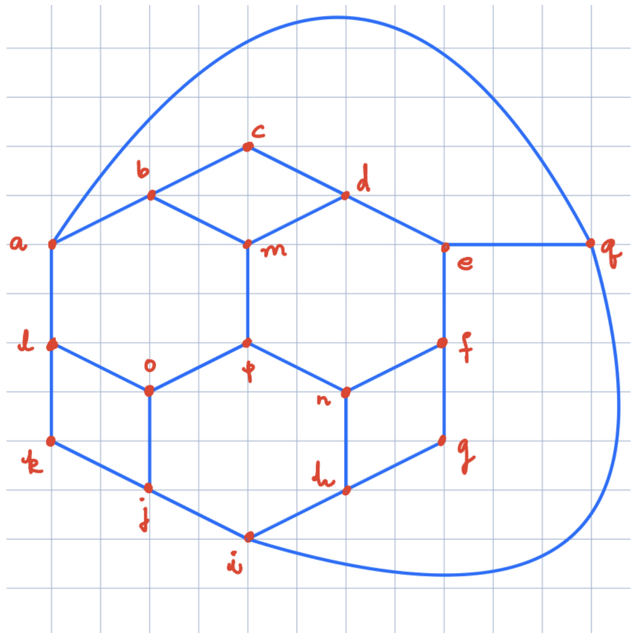


FIGURE 2. A bipartite graph with the V_1 and V_2 arranged in horizontal rows

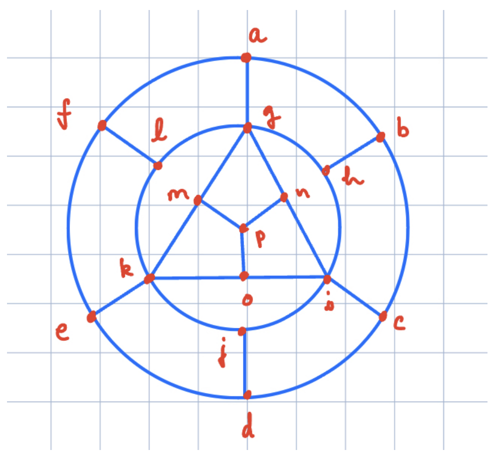
It should be pointed out that a bipartite graph need not be drawn this way. In fact it many bipartite graphs are drawn in a way that it is difficult to make out at first glance (or even subsequent glances) that they are bipartite.

1. Show that the following two graphs are bipartite by either re-drawing them according to the scheme in FIGURE 1 or the scheme in FIGURE 2. You can draw one according to one scheme and the other according to the other one, or stick to one scheme for both.

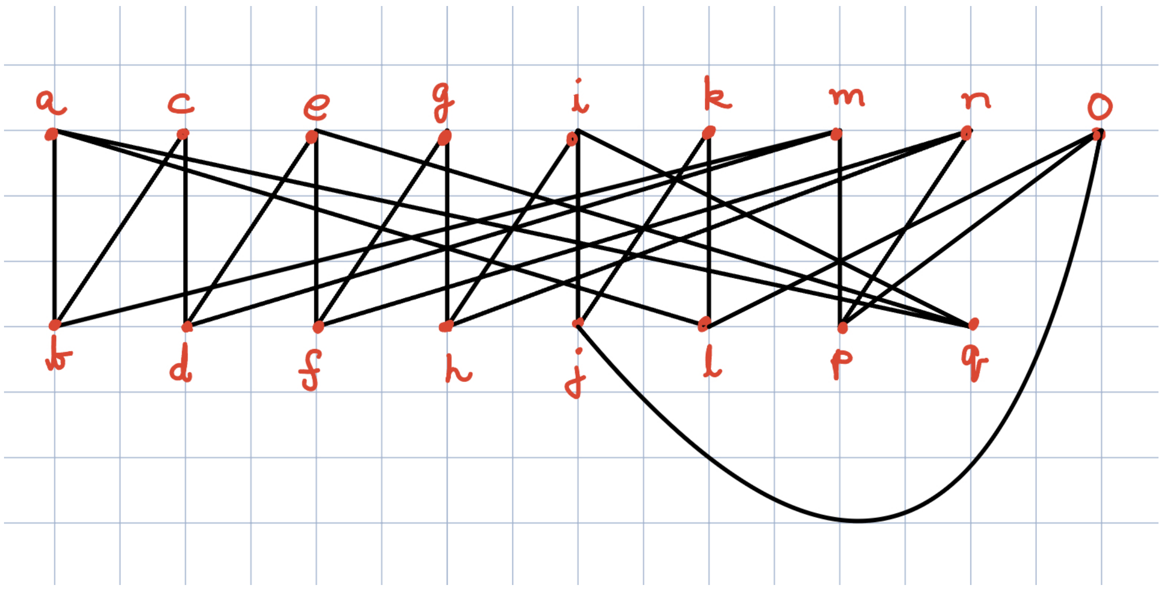
(a)



(b)

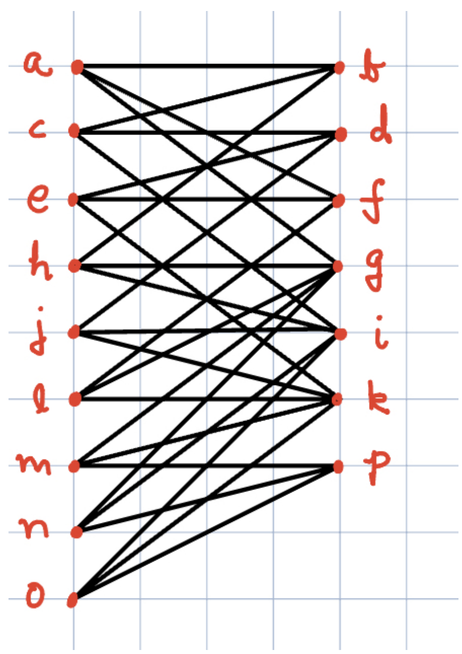


Solution: Here is a drawing of **(a)** which fits into the scheme in FIGURE 1:



There are many other possibilities for **(a)**, using either scheme. In every possibility the two disjoint non-empty subsets of V, whose union is V, are $\{a, c, e, g, i, k, m, n, o\}$ and $\{b, d, f, h, j, l, p, q\}$ and all edges are from vertices in one set to the other.

And here is one for **(b)** (this time using the scheme in FIGURE 2):



You are of course free to use the scheme in FIGURE 1 too. The partition of the set of vertices is $\{a, c, e, h, j, l, m, n, o\}$ and $\{b, d, f, g, i, k, p\}$. □

2. Let $G = (V, E)$ be a bipartite graph with V_1 and V_2 being the splitting of V given in the definition of a bipartite graph.

(a) Show that if G is hamiltonian then $|V_1| = |V_2|$.

(b) Use part **(a)** to show that the graphs in Problem **1** are not hamiltonian.

Solution: Suppose $(v_1, v_2, \ldots, v_n)$ is a hamiltonian cycle in G, then (using the convention of the textbook) the v_i are distinct, $V = \{v_1, \ldots, v_n\}$, and $v_n v_1$ is an edge. Since V is bipartite, V can be split into a union two disjoint non-empty subsets such that every edge in G is from a vertex in one of these subsets to a vertex in the other subset. Denote by V_1 the subset containing v_1, and by V_2 the other subset. Since $v_1 v_2$ is an edge, $v_2 \in V_2$. Clearly v_i is in V_1 if i is odd, and is in V_2 if i is even. Since $v_n v_1$ is an edge, we have $v_n \in V_2$, which means n is even, say $n = 2k$ for some $k \in \mathbf{N}$. From what we said, $v_{2i-1} \in V_1$, for $i = 1, \ldots, k$, and $v_{2i} \in V_1$ for $i = 1, \ldots, k$. Thus $|V_1| = k = |V_2|$. This proves **(a)**.

For part **(b)**, in the first graph in **1**, which is bipartite, we can take $V_1 = \{a, c, e, g, i, k, m, n, o\}$ and $V_2 = \{b, d, f, h, j, l, p, q\}$. Since $|V_1| = 9$ and $|V_2| = 8$, we have $|V_1| \neq |V_2|$, and so this is not a hamiltonian graph. For the second graph, which too is bipartite, we have $V_1 = \{a, c, e, h, j, l, m, n, o\}$ and $V_2 = \{b, d, f, g, i, k, p\}$, and again it is clear that $|V_1| \neq |V_2|$ since $9 \neq 7$, and hence this graph is not hamiltonian either. □

Complete graphs. A complete graph is a graph which has an edge joining any two distinct vertices. It is clear that a complete graph is connected, and any two complete graphs with the same number of vertices are isomorphic. You may use these facts in the problems that follow. We denote by $\mathbf{K}_n$ the complete graph whose set of vertices V is $[n]$, i.e. $V = \{1, 2, \ldots, n\}$.

3. Let $G = (V, E)$ be a complete graph.

(a) Show that G is hamiltonian. Note that it is enough to show that $\mathbf{K}_n$ is hamiltonian.

(b) Suppose $|V| = n$. How many edges does G have?

Solution: We first do **(a)**. If $n = 1$ this is clear, since a single vertex connected graph is considered hamiltonian. Otherwise, in $\mathbf{K}_n$, consider the path $(v_1, \ldots, v_n)$ where $v_i = i$, $i = 1, \ldots, n$. This is clearly a hamiltonian cycle.

For part **(b)** we note that in $\mathbf{K}_n$ every vertex has degree $n - 1$, and there are n vertices. Hence $\sum_{v \in V} \deg(v) = n(n-1)$. It follows that $|E| = n(n-1)/2$. There are many other solution to this problem. For example, since any two vertices are adjacent, an edge is the same as a choice of two distinct vertices, and there are $\binom{n}{2} = n(n-1)/2$ such choices. □

4. A graph G has 70 edges. What is the minimal number of vertices possible in G?

Solution: Let μ be the minimum asked for.

Since $\mathbf{K}_n$ has $\mathbf{K}_{n-1}$ as a subgraph for $n \geq 2$ (this is clear from the definition), we must have $n(n-1)/2$ is an increasing function of n as n varies over $\mathbf{N}$.

Amongst all graphs with n vertices, the one with the largest number of edges is the complete graph $\mathbf{K}_n$, since every possible choice of two distinct vertices forms an edge. Therefore if $H = (W, F)$ is a graph with $|W| = n$, then $|F| \leq n(n-1)/2$. It follows that if $n \leq 12$, then $|F| \leq n(n-1)/2 \leq 12(12-1)/2$, for, as we argued above, $n(n-1)/2$ is an increasing function of $n \in \mathbf{N}$. In other words, $|F| \leq 66$.

Thus no graph with 12 or fewer vertices can have more 66 edges. Since our graph G has 70 edges, from what we just argued, $|V| > 12$. In particular $\mu > 12$, for G is an arbitrary graph with 70 edges.

Let $G = (V, E)$ be the graph with exactly 13 vertices $v_1, \ldots, v_{13}$, with E being the set, $E = \{v_j v_j \mid 1 \leq i < j \leq 12,\ v_i v_j\} \cup \{v_1 v_{13}, v_2 v_{13}, v_3 v_{13}, v_4 v_{13}\}$. Then the subgraph with v_{13} deleted (along with all edges incident on v_{13}) is the complete graph $\mathbf{K}_{12}$. It is clear that $|E| = 66 + 4 = 70$. It follows that $\mu \leq 13$. Since $\mu > 12$, we have $\mu = 13$. $\square$

Planar graphs. Recall that a planar graph is a one which has a drawing (on the plane) such that no two edges cross each other (see Lecture 12). In class we showed that $\mathbf{K}_5$ is not planar.

5. Show that $\mathbf{K}_n$ is not planar for $n \geq 5$.

Solution: We will use the fact that a necessary condition for a connected graph $G = (V,\ E)$ (with $|V| \geq 3$) to be planar is that $3|V| - 6 \geq |E|$ (see Lecture 12, p.12 or Theorem 5.33, Chapter 5, § 5.5 of the textbook). In our case $|V| = n$ and $|E| = n(n-1)/2$ and so our strategy is to show that $3n - 6 < n(n-1)/2$ for $n \geq 5$. In other words, after re-arranging the terms in the inequality, it is sufficient for us to prove that

$$(*) \qquad n^2 - 7n + 12 > 0,$$

for $n \geq 5$.

Let $f\colon \mathbf{R} \to \mathbf{R}$ be the function given by $f(x) = x^2 - 7x + 12$, $x \in \mathbf{R}$. Its derivative is $f'(x) = 2x - 7$ which is positive for $x > 7/2$. This means f is strictly increasing in the interval $(3.5, \infty)$, in particular in $[5,\ \infty)$. Now $f(5) = 2 > 0$ and hence $f(n) \geq f(5) = 2 > 0$ for $n \geq 5$. Thus $(*)$ holds for $n \geq 5$. It follows $\mathbf{K}_n$ is not planar when $n \geq 5$. $\square$