MAT344 FALL 2022
PROBLEM SET 3

Due date: Nov 6, 2022 (on Crowdmark by midnight)

In what follows **N**, **Z**, **R** denote the set of positive integers, the set of integers, and the set of real numbers respectively. $\mathbf{N}_0$ denotes the set of non-negative integrs.

Bipartite graphs. A graph $G = (V, E)$ is said to be *bipartite* if there are two non-empty subsets V_1 and V_2 such that $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = V$ and every edge joins a vertex in V_1 to a vertex in V_2. It is easy to see that G is bipartite if and only if every walk of the form $(x_1, \ldots, x_n)$ with $x_n = x_1$ has even length. You don't have to submit a proof of this (easy) fact, but you might try and work out a proof for yourself. One typically tries to draw the vertices of V_1 on the left and V_2 on the right as below:

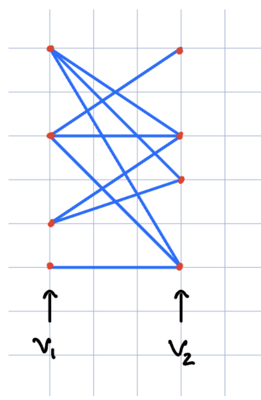


FIGURE 1. A bipartite graph drawn with V_1 on the left and V_2 on the right

Sometimes it is easier (and uses less paper) to draw it as in FIGURE 2.

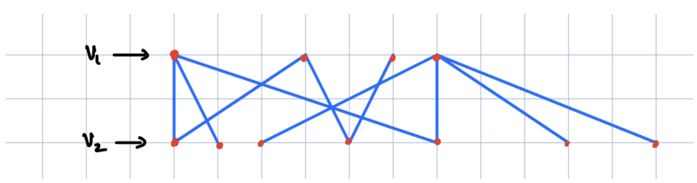
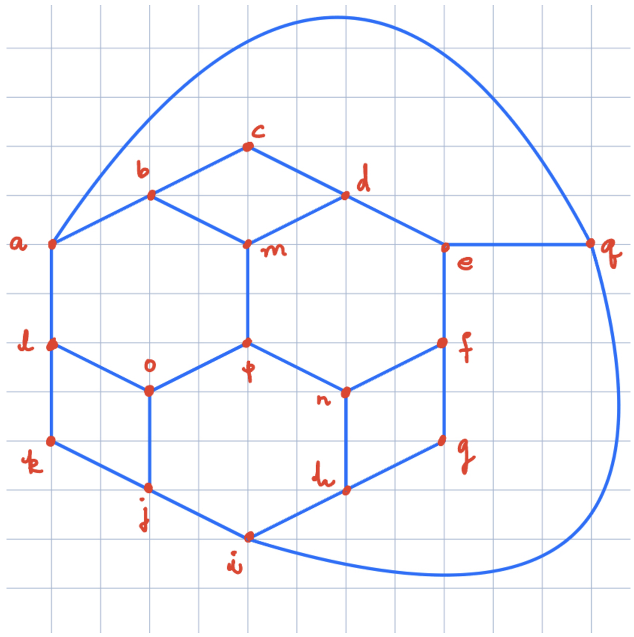


FIGURE 2. A bipartite graph with the V_1 and V_2 arranged in horizontal rows

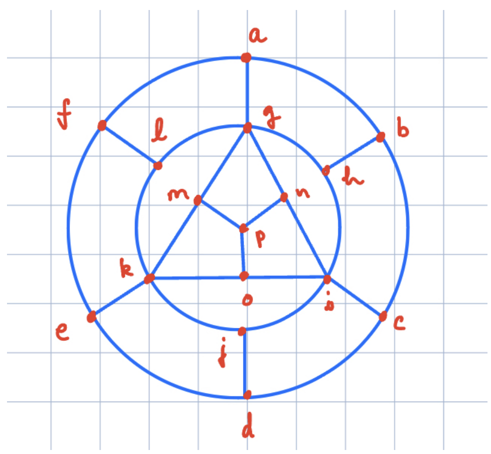
It should be pointed out that a bipartite graph need not be drawn this way. In fact it many bipartite graphs are drawn in a way that it is difficult to make out at first glance (or even subsequent glances) that they are bipartite.

1. Show that the following two graphs are bipartite by either re-drawing them according to the scheme in FIGURE 1 or the scheme in FIGURE 2. You can draw one according to one scheme and the other according to the other one, or stick to one scheme for both.

(a)



(b)



2. Let $G = (V, E)$ be a bipartite graph with V_1 and V_2 being the splitting of V given in the definition of a bipartite graph.

(a) Show that if G is hamiltonian then $|V_1| = |V_2|$.

(b) Use part (a) to show that the graphs in Problem 1 are not hamiltonian.

Complete graphs. A complete graph is a graph which has an edge joining any two distinct vertices. It is clear that a complete graph is connected, and any two complete graphs with the same number of vertices are isomorphic. You may use these facts in the problems that follow. We denote by $\mathbf{K}_n$ the complete graph whose set of vertices V is $[n]$, i.e. $V = \{1, 2, \ldots, n\}$.

3. Let $G = (V, E)$ be a complete graph.

- **(a)** Show that G is hamiltonian. Note that it is enough to show that $\mathbf{K}_n$ is hamiltonian.
- **(b)** Suppose $|V| = n$. How many edges does G have?

4. A graph G has 70 edges. What is the minimal number of vertices possible in G?

Planar graphs. Recall that a planar graph is a one which has a drawing (on the plane) such that no two edges cross each other (see Lecture 12). In class we showed that $\mathbf{K}_5$ is not planar.

5. Show that $\mathbf{K}_n$ is not planar for $n \geq 5$.