MAT344 FALL 2022
PROBLEM SET 2

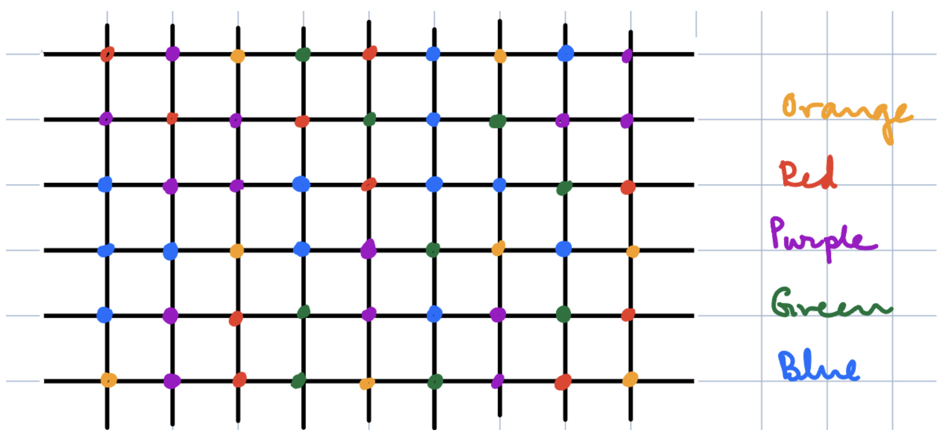
Due date: Oct 16, 2022 (on Crowdmark by midnight)

In what follows $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ denote the set of positive integers, the set of integers, and the set of real numbers respectively. $\mathbf{N}_0$ denotes the set of non-negative integrs.

1. Suppose we have a grid of 6 equidistant horizontal lines and 76 equidistant vertical lines. Colour each of the intersection points with one of five colours. Show that there is at least one rectangle (not necessarily of size 1×1) in the grid with all four vertices of the same colour.

The picture below illustrates a possible colouring scheme.

Hint: Use the pigeonhole principle multiple times, including the generalised version.



Solution: By *grid points* we mean the intersection points in the grid. Let C be the set of colours used for colouring the grid points.[1] Clearly that $|C| = 5$. Let V be the set of vertical lines in the grid and H the set of horizontal lines in the grid. We have to find a set of two distinct vertical lines $A = \{\ell, \ell'\} \subset V$, and a set of two distinct horizontal lines $B = \{h, h'\} \subset H$, such that the four intersection points defined by $\ell \cap h$, $\ell \cap h'$, $\ell' \cap h$, and $\ell' \cap h'$ have the same colour.

If $\ell \in V$ is a vertical line, then by the Pigeon Hole principle there is (at least) one set of distinct horizontal lines $B(\ell) = \{h(\ell), h'(\ell)\}$ such that the two points defined by the two ntersections $h(\ell) \cap \ell$ and $h'(\ell) \cap \ell$, have the same colour, say c_ℓ. Define a map

$$f : V \to C$$

[1] In the picture $C = \{\text{Orange, Red, Purple, Green, Blue}\}$.

by the formula $f(\ell) = c_\ell$. Now

$$|V| = 76 = (15)\cdot 5 + 1 = (16-1)\cdot |C| + 1 > (16-1)\cdot |C|.$$

By the Strong Pigeon Hole Principle, we have a colour $c \in C$ and 16 vertical lines, $\ell_1, \ldots, \ell_{16}$, such that $f(\ell_1) = f(\ell_2) = \cdots = f(\ell_{16}) = c$.

Now there are $\binom{6}{2} = 15$ possibilities for the set $B = \{h, h'\}$ mentioned in the first paragraph. This means, once again by the Pigeon Hole Principle, at least two of the 16 sets $B(\ell_1), \ldots, B(\ell_{16})$, are the same, say $B(\ell_m) = B(\ell_n)$, with $m \neq n$. Let $A = \{\ell_m, \ell_n\}$ and $B = B(\ell_m) = B(\ell_n)$. Then A and B are our required sets. The four vertices defined by them are all coloured c. $\square$

2. Give a combinatorial proof of the following identity

$$7^n - 6^n = \sum_{i=1}^{n} 6^{i-1}7^{n-i}.$$

Hint: Consider strings of length n from the set $\{0, 1, 2, 3, 4, 5, 6\}$.

Solution: The left side counts the number of strings of length n from the set $\{0, 1, 2, \ldots, 6\}$ which contain at least one 0. The i^{th} summand on the right side is the number of strings where 0 occurs for the first time at the i^{th} place. $\square$

3. Let $n \in \mathbf{N}$. Give a proof using lattice paths of the identity

$$\binom{2n}{n} = \binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{2n-1}{n}.$$

The choice of the lattice path type (usual or diagonal) is left to you. It might be simpler to use usual lattice paths for this problem rather than the diagonal ones.

Solution: Let S be the set of lattice paths from $(0,\, 0)$ to $(n,\, n)$. For $0 \le j \neq n$, let $S_j = \{\sigma \in S \mid \text{the first time } \sigma \text{ hits the vertical line } x = n \text{ is at } (n,\, j)\}$. Then S_j are pairwise disjoint and $\cup_{j=0}^{n} S_j = S$. Thus

$$(*) \qquad |S| = \sum_{j=0}^{n} |S_j|.$$

There is only one lattice path from (n, j) to (n, n). (This is clear since lattice paths are not allowed to move left. One can also prove the assertion by using the formula $\binom{n-j}{0} = \binom{n-j}{n-j} = 1$.)

Note also that a lattice path from $(0, 0)$ to (n, j) hits the line $x = n$ for the first time at (n, j) if and only if in the previous step it was at $(n-1, j)$.

Therefore the number of paths in S_j is the same as the number of lattice paths from $(0, 0)$ to $(n-1,\, j)$. Thus $|S_j| = \binom{n+j-1}{j}$. From $(*)$ we therefore get

$$\binom{2n}{n} = \sum_{j=0}^{n} \binom{n+j-1}{j},$$

which is what we were required to prove. $\square$

4. Let n and k be integers such that $n \geq 2$ and $1 \leq k \leq n$. Let S be the set of *diagonal lattice paths* from (k, k) to $(2n, 0)$ which pass through the point $(2n-k, k)$. How many paths in S are such that they touch the line $y = k - 1$ for the first time at the point $(2n - k + 1, k - 1)$?

Solution: It is easy to see that there is only one diagonal lattice path from the point $(2n - k + 1, k - 1)$ to $(2n, 0)$. If the first time a diagonal lattice path σ from (k, k) hits $y = k - 1$ is at $(2n - k + 1, k - 1)$, then σ must be at $(2n - k, k)$ in the previous step. Thus the problem amounts to finding the number of diagonal lattice paths from $(k\,k)$ to $(2n-k,\ k)$ which never dip below the line $y = k$. This is clearly the same as the number of diagonal lattice paths from $(0,\ 0)$ to $(2(n-k),\ 0)$ which never dip below the x-axis. Thus the answer is

$$C_{n-k} = \frac{1}{n-k+1}\binom{2(n-k)}{n-k}.$$

□

5. Let $n \geq 3$. Use induction to prove that the sum of the internal angles of a convex n-gon is $180(n-2)$.

Solution: This is straightforward. It is well known for triangles. If $n \geq 4$, label the vertices $v_1, \ldots, v_n$ so that v_i is adjacent to v_{i+1} and v_n is adjacent to v_1. Draw a line from v_1 to v_3 to break up the convex n-gon into an $(n-1) - gon$ and a triangle. Apply the induction hypothesis and the base case (i.e. the triangle case). □

6. Let $n \in \mathbf{N}$. Consider the series

$$f(n) = \frac{1}{1 \cdot 2} + + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)}$$

Experiment with a few values of n and guess a formula for $f(n)$. Prove your conjecture using induction.

Solution: Easy to guess that $f(n) = \dfrac{n}{n+1}$, $n \in \mathbf{N}$. A straightforward induction proves this. □