MAT344 FALL 2022
PROBLEM SET 2

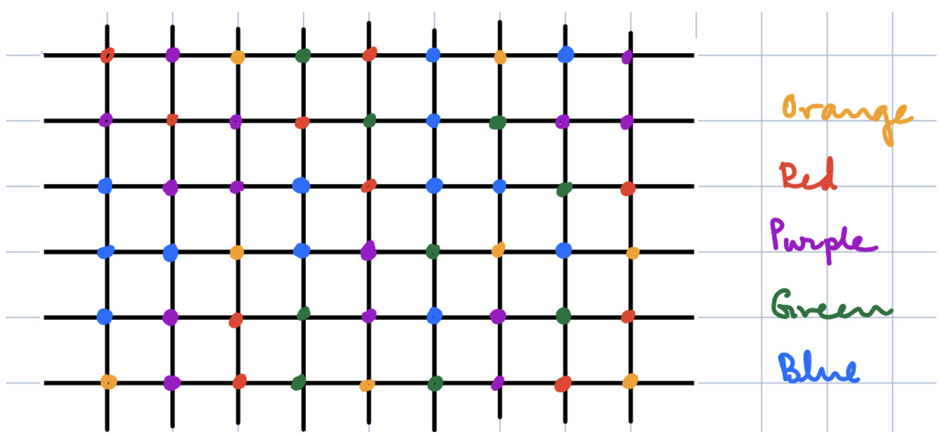
Due date: Oct 16, 2022 (on Crowdmark by midnight)

In what follows **N**, **Z**, **R** denote the set of positive integers, the set of integers, and the set of real numbers respectively. $\mathbf{N}_0$ denotes the set of non-negative integrs.

1. Suppose we have a grid of 6 equidistant horizontal lines and 76 equidistant vertical lines. Colour each of the intersection points with one of five colours. Show that there is at least one rectangle (not necessarily of size 1×1) in the grid with all four vertices of the same colour.

The picture below illustrates a possible colouring scheme.

Hint: Use the pigeonhole principle multiple times, including the generalised version.



2. Give a combinatorial proof of the following identity

$$7^n - 6^n = \sum_{i=1}^{n} 6^{i-1} 7^{n-i}.$$

Hint: Consider strings of length n from the set $\{0, 1, 2, 3, 4, 5, 6\}$.

3. Let $n \in \mathbf{N}$. Give a proof using lattice paths of the identity

$$\binom{2n}{n} = \binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{2n-1}{n}.$$

The choice of the lattice path type (usual or diagonal) is left to you. It might be simpler to use usual lattice paths for this problem rather than the diagonal ones.

4. Let n and k be integers such that $n \geq 2$ and $1 \leq k \leq n$. Let S be the set of *diagonal lattice paths* from (k, k) to $(2n, 0)$ which pass through the point $(2n-k, k)$. How many paths in S are such that they touch the line $y = k-1$ for the first time at the point $(2n-k+1, k-1)$?

5. Let $n \geq 3$. Use induction to prove that the sum of the internal angles of a convex n-gon is $180(n-2)$.

6. Let $n \in \mathbf{N}$. Consider the series

$$f(n) = \frac{1}{1 \cdot 2} + + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)}$$

Experiment with a few values of n and guess a formula for $f(n)$. Prove your conjecture using induction.