MAT344 FALL 2022
PROBLEM SET 1

Due date: Oct 2, 2022 (on Crowdmark by midnight)

In what follows $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ denote the set of positive integers, the set of integers, and the set of real numbers respectively. $\mathbf{N}_0$ denotes the set of non-negative integers.

1. How many arrangements are there of the 7 letters in ADAMANT?

Solution: There are two ways of doing this problem.

First Way. Pick three spots out of seven to slot the A's in, and then distribute D, M, N, T into the remaining four spots, in any way you can. This accounts for $\binom{7}{3}(4!) = \frac{7!}{3!}$ number of ways.

Second Way. There are 7! ways of permuting 7 objects. However, for any arrangement of the letters, the three A's can be arbitrarily permuted amongst themselves to give the same arrangement. We once again get $\binom{7}{3}(4!)$ as the answer. □

2. Let $k \geq n$. Give a **combinatorial proof** of the identity

$$\sum_{r=0}^{n} \binom{n}{r}\binom{k}{r} = \binom{n+k}{n}.$$

Solution: Split S into two disjoint sets: $A = [n]$ and $B = \{n+1, \ldots, n+k\}$. Picking n elements from S is the same as picking k elements from S, or what is the same thing, picking a subset C of S with $|C| = k$. Let $r = |C \cap A|$. Then $k - r = |C \cap B|$. Now r can be any number in $\{0, 1, \ldots, n\}$. For a fixed $r \in \{0, 1, \ldots, n\}$ there are $\binom{n}{r}$ ways of picking $C \cap A$, and $\binom{k}{k-r}$ ways of picking $C \cap B$. It follows that there are $\sum_{r=0}^{n} \binom{n}{r}\binom{k}{r}$ ways of picking C such that $|C| = k$. In other words $\binom{n+k}{k} = \sum_{r=0}^{n} \binom{n}{r}\binom{k}{r}$. Since $\binom{n+k}{k} = \binom{n+k}{n}$, we are done. □

3. Let $n \geq 9$ be an integer. Give a combinatorial proof of the identity

$$\binom{n}{9} = \sum_{k=4}^{n-5} \binom{k-1}{3}\binom{n-k}{5}.$$

Solution: The left side counts the number of subsets A of $[n]$ such that $|A| = 9$. Let A be such a set and write the elements of A in ascending order as $x_1 < \cdots < x_9$. Then $k = x_4$ must be such that $0 \leq k \leq n-5$. Conversely, given $k \in \{4, 5, \ldots, n-5\}$, there are $\binom{k-1}{3}$ ways of picking three elements in $[k-1]$, written in ascending order as $x_1 < x_2 < x_3$, and $\binom{n-k}{5}$ ways of picking five elements from $\{k+1, k+2, n\}$, written in ascending order as $x_5 < x_6 < x_7 < x_8 < x_9$. Setting $k = x_4$ the set $A = \{x_1, \ldots, x_9\}$ has size 9. Thus $\binom{n}{9} = \sum_{k=4}^{n-5} \binom{k-1}{3}\binom{n-k}{5}$. $\square$

4. In how many ways can you distribute 12 identical objects to 5 people if the first person can only have 4 or 5 objects and the second person cannot have more than 3 objects?

Solution: Let us first solve the problem without the constraint on the second person. The constraint on the first person amounts to finding the sum of the number of non-negative integer solutions of $x_2 + x_3 + x_4 + x_5 = 8$ and the number of non-negative integer solutions of $x_2 + x_3 + x_4 + x_5 = 7$. From this we have to subtract the number of solutions which violate the second constraint, i.e. the ones for which $x_2 \geq 4$. In other words we have to remove the $\binom{7}{3}$ solutions of $x_2+x_3+x_4+x_5 = 8$ such that $x_3 + x_4 + x_5 \leq 4$, as well as the $\binom{6}{3}$ solutions of $x_2 + x_3 + x_4 + x_5 = 7$ such that $x_3 + x_4 + x_5 \leq 3$. The answer is $\binom{11}{3} + \binom{10}{3} - \binom{7}{3} - \binom{6}{3}$. $\square$

Diagonal Lattice Paths. Recall that there is another kind of lattice path, the so-called *diagonal lattice path*, that is common in the literatute. These consist of steps from (i, j) to either $(i + 1, j + 1)$ or $(i + 1, j - 1)$. Recall that the number of diagonal lattice maths from $(0, 0)$ to (m, n) is $C(m, (m - n)/2)$ where our convention is that for a real number r, $C(m, r) = 0$ if r does not belong to $\{0, 1, \ldots, m\}$.

5. Let $n \in \mathbf{N}$. Assume that there is a diagonal lattice path from $(0, 0)$ to $(n, 3)$. Prove combinatorially that the number of diagonal lattice paths from $(0, 0)$ to $(n, 3)$ which dip below the line $y = -1$ is $C(n, (n + 7)/2)$.

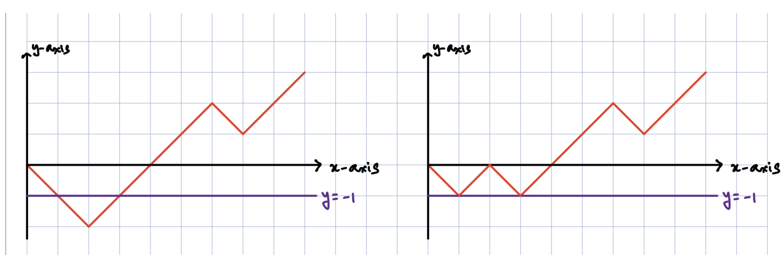

FIGURE 1. The path on the left dips below the line $y = -1$ but the path on the right does not.

Solution: For brevity, by a path we will mean a diagonal lattice path (for this solution). Let S be the set of paths from $(0, 0)$ to $(n, 3)$. A path dips below the line $y = -1$ if and only if it hits the line $y = -2$. Let $\sigma \in S$, and let $k = k$ be the first time σ hits the line $y = -2$. Let σ_1 be the portion of σ between $(0, 0)$ and $(k, -2)$ and σ_2 the portion between $(k, -2)$ and $(n, 3)$. Reflect σ_2 about the line $y = -2$ and get a path τ_2. It is clear that τ_2 is a path from $(k, -2)$ to $(n, -7)$, since -2 is exactly midway between 3 and -7. Let τ be the lattice path from $(0, 0)$ to $(n, -17)$ which is σ_1 followed by τ_2. Thus σ gives rise to a path τ from $(0, 0)$ to $(n, -17)$.

Conversely, if τ is a path from $(0, 0)$ to $(n, -17)$, it must hit the line $y = -2$. Let k be the first time it does, and break up τ into τ_1 and τ_2, with τ_1 being the portion of τ on or before $(k,\, -2)$, and τ_2 the portion from $(k, -2)$ to $(n, -7)$. Reflect τ_2 about the line $y = -2$ to get a path σ_2 from $(k, -2)$ to $(n, 3)$. Let σ be the path which is τ_1 followed by σ_2. It is clear that $\sigma \in S$ since it must dip below $y = -1$ if it has the point $(k,\, -2)$ on it.

The two processes are clearly inverses of each other and give a bijective correspondence between S and the set of paths from $(0,\, 0)$ to $(n,\, -7)$. The number of paths from $(0,\, 0)$ to $(n, -7)$ is $C(n, (n + 7)/2)$ and hence we are done. $\square$