MAT344 FALL 2022
PRACTICE QUESTIONS FOR THE FINAL

In what follows $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ denote the set of positive integers, the set of integers, and the set of real numbers respectively. $\mathbf{N}_0$ denotes the set of non-negative integers.

Counting problems and induction.

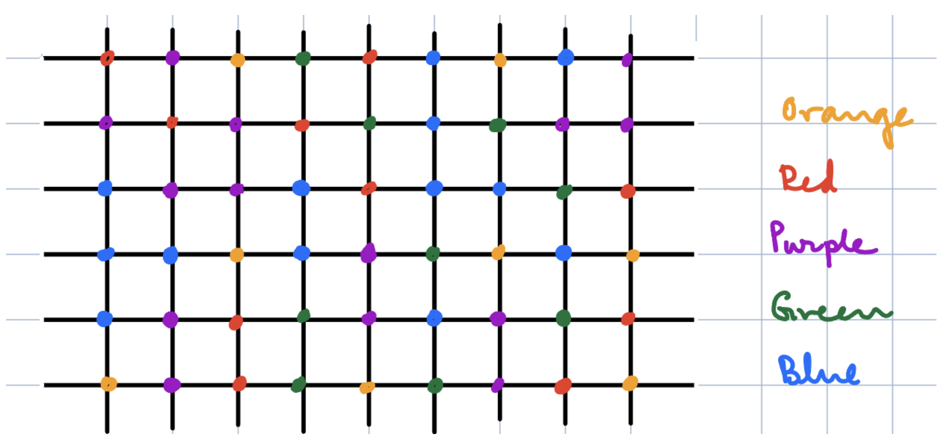
1. How many arrangements are there of the 7 letters in ADAMANT?

2. In how many ways can you distribute 12 identical objects to 5 people if the first person can only have 4 or 5 objects and the second person cannot have more than 3 objects?

3. Suppose we have a grid of 6 equidistant horizontal lines and 76 equidistant vertical lines. Colour each of the intersection points with one of five colours. Show that there is at least one rectangle (not necessarily of size 1×1) in the grid with all four vertices of the same colour.

The picture below illustrates a possible colouring scheme.

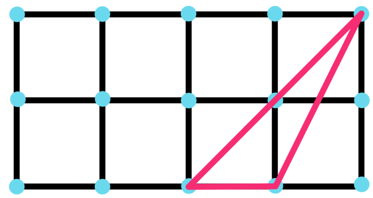
Hint: Use the pigeonhole principle multiple times, including the generalised version.



4. In how many ways can we sit 12 people around a round table if three friends insist of sitting together? Configurations that can be identified by rotation are considered the same.

5. How many triangles with vertices placed at the intersection points of a 2×4 square grid are there? The picture below shows an example; the intersection points are the 15 blue dots.

Note. A configuration with three points in a line does not count: the triangle must have positive area.)



6. Let $n \geq 3$. Use induction to prove that the sum of the internal angles of a convex n-gon is $180(n-2)$.

7. Let $n \in \mathbf{N}$. Consider the series

$$f(n) = \frac{1}{1 \cdot 2} + + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)}$$

Experiment with a few values of n and guess a formula for $f(n)$. Prove your conjecture using induction.

8. Consider the set $[n] = \{1, 2, \ldots, n\}$.

(a) Show that the number of functions $f \colon [n] \to S$, with $|S| = d$ is d^n.

(b) Show that there are 6^n ways of finding 5 subsets A_1, A_2, A_3, A_4, and A_5 of $[n]$ such that $A_1 \subset A_2 \subset A_3 \subset A_4 \subset A_5$.

(c) How many of the ways in **(b)** are such that $A_3 \neq A_4$?

Combinatorial proofs of identities.

9. Let $m, n \in \mathbb{Z}$ with $0 \leq m \leq n$. Give a ***combinatorial proof*** of the equality

$$\sum_{i=0}^{n-m} \binom{n}{i} \binom{n-i}{n-m-i} = \binom{n}{m} 2^{n-m}.$$

10. Let $k \geq n$. Give a **combinatorial proof** of the identity

$$\sum_{r=0}^{n} \binom{n}{r} \binom{k}{r} = \binom{n+k}{n}.$$

11. Give a combinatorial proof of the following identity

$$7^n - 6^n = \sum_{i=1}^{n} 6^{i-1} 7^{n-i}.$$

Hint: Consider strings of length n from the set $\{1, 2, 3, 4, 5, 6, 7\}$.

12. Give a combinatorial proof of the following equality

$$7^n - 5^n = 2 \sum_{i=1}^{n} 5^{i-1} 7^{n-i}.$$

13. Let $n \geq 9$ be an integer. Give a combinatorial proof of the identity

$$\binom{n}{9} = \sum_{k=4}^{n-5} \binom{k-1}{3}\binom{n-k}{5}.$$

Lattice Paths.

14. Let $n \in \mathbf{N}$. Assume that there is a diagonal lattice path from (0, 0) to $(n, 3)$. Prove combinatorially that the number of diagonal lattice paths from $(0, 0)$ to $(n, 3)$ which dip below the line $y = -1$ is $C(n, (n + 7)/2)$.

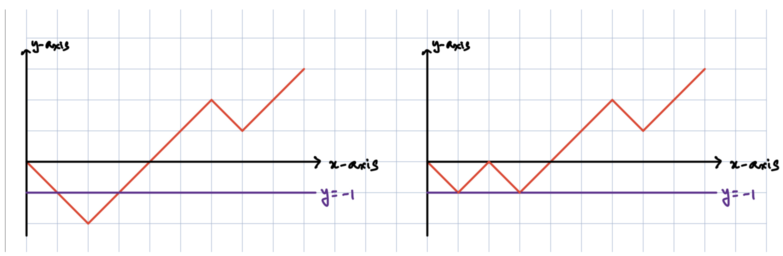


FIGURE 1. The path on the left dips below the line $y = -1$ but the path on the right does not.

15. Let $n \in \mathbf{N}$. Give a proof using lattice paths of the identity

$$\binom{2n}{n} = \binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{2n-1}{n}.$$

The choice of the lattice path type (usual or diagonal) is left to you. It might be simpler to use usual lattice paths for this problem rather than the diagonal ones.

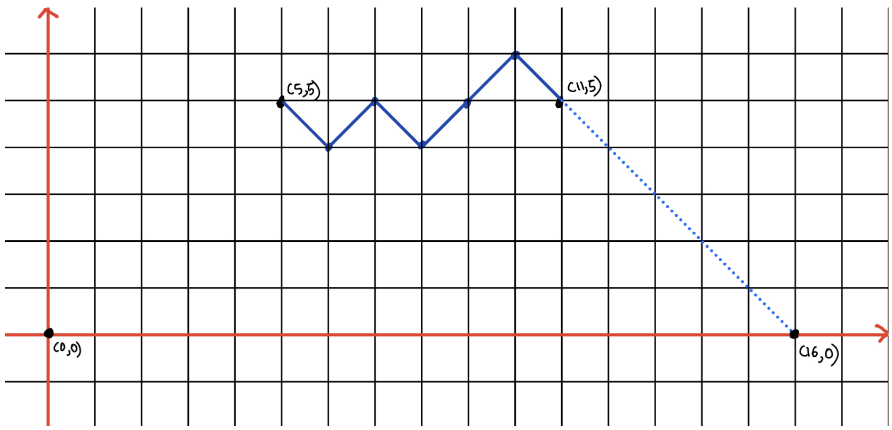
16. Let n and k be integers such that $n \geq 2$ and $1 \leq k \leq n$. Let S be the set of *diagonal lattice paths* from (k, k) to $(2n, 0)$ which pass through the point $(2n - k, k)$. How many paths in S are such that they touch the line $y = k - 1$ for the first time at the point $(2n - k + 1, k - 1)$?

17. Give a lattice path proof of

$$\binom{3n}{2n} = \sum_{k=0}^{n} \binom{n+k}{n}\binom{2n-k-1}{n-1}, \qquad n \in \mathbf{N}.$$

18. How many diagonal lattice paths are there from (5, 5) to (16, 0) which pass through the point (11, 5)?

19. Give lattice path proofs for all the identities in Problems **9**, **10**, **11**, **12**, and **13**.



Graph Theory.

20. Suppose a graph $G = (V, E)$ is such that every cycle is of even length. Show that V can be written as $V = V_1 \cup V_2$, with $V_1 \cap V_2 = \emptyset$, such that every edge joins a vertex in V_1 to a vertex in V_2.

21. Let $G = (V, E)$ be a bipartite graph with V_1 and V_2 being the splitting of V given in the definition of a bipartite graph. Show that if G is hamiltonian then $|V_1| = |V_2|$.

22. Let $G = (V, E)$ be a complete graph.
  (a) Show that G is hamiltonian. Note that it is enough to show that $\mathbf{K}_n$ is hamiltonian.
  (b) Suppose $|V| = n$. How many edges does G have?

23. A graph G has 70 edges. What is the minimal number of vertices possible in G?

24. Show that $\mathbf{K}_n$ is not planar for $n \geq 5$.

25. Let $G = (V, E)$ be a graph such that $2 \leq |V| < \infty$. Suppose $2\deg_G(v) \geq |V| - 1$ for all $v \in V$. Show that G is connected. **Hint:** Try a proof by contradiction.

26. Let $G = (V, E)$ be a grap and suppose $|V| = 26$. Show that if G has three connected components, then there is at least one component with which has fewer than 9 vertices. Conclude that there is at least one vertex v such that $\deg_G(v) \leq 7$.

Inclusion-Exclusion.

27. Let X be a finite set and $A_1, A_2, \ldots, A_n$ subsets of X. For any subset T of $[n]$, let $A_T = \bigcap_{j \in T} A_j$, with the understanding that if $T = \emptyset$, then $A_T = X$. Fix a subset S of $[n]$. Let Y be the subset of X consisting of all elements of X which belong to A_i for every $i \in S$, but for no other indices. In other words,

$$Y = \{x \in A_S \mid x \notin A_i \text{ if } i \notin S\}.$$

Show that

$$|Y| = \sum_{S \subset T \subset [n]} (-1)^{|T \setminus S|} |A_T|.$$

The sum is taken over subsets T of $[n]$ which contain S. **Note:** The symbol $\subset$ denotes inclusion. For strict inclusion we use the symbol $\subsetneq$.

28. Let S be the set of maps $f\colon [10] \to [7]$ such that $[4] \subset f([10])$. Show using the Inclusion-Exclusion Principle that

$$|S| = \sum_{k=0}^{4} \binom{4}{k} (-1)^k (7-k)^{10}.$$

Generating functions.

29. Find the number of $\{A, B, C, D\}$-strings of length $n \in \mathbf{N}$ containing an odd number of As, an even number of Bs, and at least one C.

30. For $n \in \mathbf{N}_0$, let a_n be the number of non-negative integer solutions of

$$3a + b + 7c + d = n$$

with the added conditions that $b \leq 2$, $c \geq 1$, and $d \leq 6$. Find the generating function for $(a_n)_{n=0}^{\infty}$. Use this to determine a_n for $n \in \mathbf{N}_0$.

31. Show using exponential generating functions that the number of surjective maps from $[n]$ to $[m]$ is $\sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n$. We proved this in Lecture 14. Conclude that if $n < m$ then $\sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n = 0$. Check this is true for small values of n and m, for example $n = 3$ and $m = 5$.

Recurrence relations.

32. For all $n \in \mathbf{N}$, let a_n be the number of ternary strings of length n where no two consecutive digits are non-zero. For example, if $n = 8$, then 01000201 and 10201020 are valid, but 01002100 is not valid since 21 are two consecutive digits which are non-zero. Also, 10200110 is not valid because of 11.

(a) Find a recurrence relation for a_n.

(b) Solve for a_n using the method of advancement operators.

Note: Since $n \in \mathbf{N}$, there is no a_0. You need to find a_1 and a_2 for your initial conditions.

33. Suppose $(a_n)_{n=0}^{\infty}$ satisfies the recursion relation $a_n = (n-1)(a_{n-1} + a_{n-2})$, $n \geq 2$, with initial conditions: $a_0 = 1$ and $a_1 = 0$. Show that $(a_n)_{n=0}^{\infty}$ also satisfies the recurrence relation

$$a_n = n a_{n-1} + (-1)^n, \qquad n \in \mathbf{N}.$$

Hint: Consider $b_n = a_n - n a_{n-1}$, $n \in \mathbf{N}$, $b_0 = 1$. Find a suitable recurrence relation for $(b_n)_{n=0}^{\infty}$ and solve that recurrence relation.

34. Let $(a_n)_{n=0}^{\infty}$ be the sequence defined by

$$a_n = 3a_{n-1} + 2a_{n-2}, \quad \text{for all } n \geq 2,$$

with initial conditions, $a_0 = 0$, and $a_1 = 1$. Use the method of generating functions to find a_n, $n \in \mathbf{N}_0$.

35. For $n \in \mathbf{N}$, let Δ_n be the number of derangements of $[n]$ and set $\Delta_0 = 1$. Recall that we proved in class that

$$\Delta_n = \sum_{k=0}^{n} (-1)^n \binom{n}{k} (n-k)! = n! \sum_{k=0}^{n} (-1)^k / k!.$$

Show combinatorially, *without using the above formulas*, that

$$\Delta_n = (n-1)(\Delta_{n-1} + \Delta_{n-2})$$

for $n \geq 2$.

Probability Theory.

36. Suppose that we have found that the word "Ferrari" occurs in 250 of 2500 messages known to be spam and in 5 of 1000 messages known not to be spam. Assume that these proportions are a good estimate for certain obvious conditional probabilities. Estimate the probability that an incoming message containing the word "Ferrari" is spam. You may assume that a message coming in has an equal chance of being spam or not spam.

37. You have three different kinds of candies, say red, blue, and green, and 10 of each kind. What is the probability that of you randomly handing out 5 blue, 2 green, and 3 red candies to a friend?

38. A string is chosen at random from the set of all binary strings of length n. Show that the expected number of zeroes in the string is $\frac{n}{2}$.

39. A group of nine people sit randomly around a round table. In the group there is a brother and sister pair. What is the probability that the brother and sister sit next to each other? (Remember our conventions from the early lectures about round table seating arrangements.)

40. In a bridge hand, you and your partner (whose hand you know) have seven clubs between the two of you. What is the probability that one of your opponents has exactly four of the remaining clubs?

41. A process sends an object along **diagonal lattice paths** on the xy-plane. This means the object only moves in the northeast or southeast directions, and can only change direction at lattice points, i.e. points with integer coordinates. Suppose the process is random and sends the object from $A = (0, 0)$ to $B = (12, 2)$ in such a way that every diagonal lattice path from A to B has an equal chance of being travelled on by the object. What is the probability that the object will travel along the segment from $(5, 3)$ to $(6, 2)$?