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STUDENT NUMBER:

University of Toronto
Faculty of Arts and Sciences

(2019)

MAT344 PRACTICE EXAM

Duration: 3 hours
Aids Allowed: None

Exam Reminders:

- Fill out your name and student number at the top of this page.
- Do not begin writing the actual exam until the announcements have ended and the Exam Facilitator has started the exam.
- As a student, you help create a fair and inclusive writing environment. If you possess an unauthorized aid during an exam, you may be charged with an academic offence.
- Turn off and place all cell phones, smart watches, electronic devices, and unauthorized study materials in your bag under your desk. If it is left in your pocket, it may be an academic offence.
- When you are done with your exam, raise your hand for someone to come and collect your exam. Do not collect your bag and jacket before your exam is handed in.
- If you are feeling ill and unable to finish your exam, please bring it to the attention of an Exam Facilitator so it can be recorded before leaving the exam hall.
- In the event of a fire alarm, do not check your cell phone when escorted outside.

Special Instructions:

- If you need scratch paper, use the back of the pages. **We will only read and grade what you write on the front of each page.**
- **If you need extra space for a question, you may use Pages 10 and 11 for this purpose.** If you do so, clearly indicate it on the corresponding problem page.
- In order to get full points, **you need your final answer to be correct and you need a justification,** unless otherwise indicated. You do not need to evaluate binomials or factorials.

Exam Format and Grading Scheme:

Answers must be written on the examination paper.

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	10	10	10	10	10	10	10	10	90

Students must hand in all examination materials at the end

PRACTICE FINAL FROM WINTER 2019 (Instructors: Henry Yuen & Zack Wolske)

(Caveat Emptor. I solved this in a hurry.)

1. (10 points) Determine the closed form expression for the n'th term of each generating function:

(a) $\frac{3x}{1-4x} + \frac{1}{1-x}$

(b) $(3+x)^6$

(c) $\frac{1}{1+x^3}$

(a) $$\frac{3x}{1-4x} + \frac{1}{1-x} = 3x\left\{\sum_{n=0}^{\infty} 4^n x^n\right\} + \sum_{n=0}^{\infty} x^n = 1 + \sum_{n=1}^{\infty}\left\{3\cdot 4^{n-1}\right\}x^n$$

$$a_n = \begin{cases} 1 & \text{if } n=0 \\ 3\cdot 4^{n-1}+1 & \text{if } n \geq 1 \end{cases}$$

(b) $a_n = \binom{6}{n} 3^{6-n}$, with the understanding that $\binom{6}{n} = 0$ if $n > 6$.

(c) $$\frac{1}{1+x^3} = \sum_{n=0}^{\infty} (-1)^n x^{3n}.$$

$$a_n = \begin{cases} 0 & \text{if } n \text{ is not a multiple of three} \\ (-1)^{n/3} & \text{if } n \text{ is a multiple of three.} \end{cases}$$

For each of the following sequences, find a closed form expression for the corresponding generating function (that does not involve an infinite sum). The sequences begin with index 0.

(c) $[1, 3, 5, 7, 9, \ldots, 2k+1, \ldots]$

(d) $[1, 4, 9, 16, 25, \ldots, k^2, \ldots]$

(c) The sequence is $(a_n)_{n=0}^{\infty}$ where $a_n = 2n+1$, $n \in \mathbb{N}_0$. The generating function is

$$f(x) = \sum_{n=0}^{\infty} (2n+1)x^n = \sum_{n=0}^{\infty}\{2(n+1)-1\}x^n = 2\sum_{n=0}(n+1)x^n - \sum_{n=0} x^n$$

$$= \frac{2}{(1-x)^2} - \frac{1}{1-x} = \frac{1+x}{(1-x)^2}.$$

Closed form: $f(x) = \frac{1+x}{(1-x)^2}$.

(d) The displayed sequence is $(a_n)_{n\geq 0}$ where $a_n = (n+1)^2$. We have:

$$(n+1)^2 = (n+1)(n+2) - (n+1) = 2\binom{n+2}{2} - \binom{n+1}{1}.$$

Thus, if $f(x)$ is the generating function of $(a_n)_{n\geq 0}$, then

$$f(x) = 2\sum_{n=0}^{\infty}\binom{n+2}{2}x^n - \sum_{n=0}^{\infty}\binom{n+1}{1}x^n = \frac{2}{(1-x)^3} - \frac{1}{(1-x)^2} = \frac{1+x}{(1-x)^3}$$

closed form

2. (10 points) Suppose there are five shows on your Netflix list. Three have 25 minute episodes, the other two have 50 minute episodes. You will watch the episodes of each show in order, but you can change shows after each episode, and watch shows in any order. Suppose you have $25n$ minutes to watch tv between studying. Let a_n be the number of ways you can watch these shows during that time.

$a_1 = 1$ is obvious. As for a_2, we have nine $\{A_1, A_2, A_3\}$- strings of length Two as well as B_1B_1 and B_2B_2. So $a_2 = 11$

(a) Give a recurrence and initial conditions for a_n.

Let $X = \{A_1, A_2, A_3, B_1, B_2\}$ where A_1, A_2, A_3 are the short shows and B_1, B_2 the long ones. For $n \in \mathbb{N}$, a $25n$-minute binge of watching is represented by an X-string of length n, with the condition that every occurrence of B_1 is part of a string of an even # of successive B_1's; and exactly the same condition applies to B_2. Let Y_n be the set of such X-strings of length n. Then $a_n = |Y_n|$.

Let $\sigma \in Y_n$. If the last character of σ is one of the A_i's, then removing this character gives us an element of Y_{n-1}. If the last character is one of the B_i's, then such a removal does not give an element of Y_{n-1}, but removing the last two gives us an element of Y_{n-2}.

Thus $a_n = 3a_{n-1} + 2a_{n-2}$, $n \geq 3$. Now $a_1 = 3$ and $a_2 = 11$ (see margin)

(b) Use your recurrence and initial conditions to find a closed formula for a_n depending only on n. (If your solution to part (a) is incorrect, you will still receive marks for analysing it correctly here.)

We have to solve $a_n = 3a_{n-1} + 2a_{n-2}$, $n \geq 3$, with $a_1 = 3$ and $a_2 = 11$. This is the same as $a_n = 3a_{n-1} + 2a_{n-2}$, $n \geq 2$, with $a_0 = 1$ and $a_1 = 3$ (check!).

The characteristic polynomial is $x^2 - 3x - 2 = (x-\alpha)(x-\beta)$ where

$$\alpha = \tfrac{1}{2}(3 + \sqrt{17}), \quad \beta = \tfrac{1}{2}(3 - \sqrt{17}) \qquad \text{(use the quadratic formula)}$$

Since $\alpha \neq \beta$, this gives

$$a_n = c\alpha^n + d\beta^n, \quad n \in \mathbb{N}_0$$

where c and d are constants such that $a_0 = 1$ and $a_1 = 3$. Setting $n = 0$ we get $c + d = 1$ and setting $n = 1$ we get $c\alpha + d\beta = 3$. This gives

$$c = \frac{1}{\sqrt{17}}\left(\frac{3+\sqrt{17}}{2}\right) \text{ and } d = \frac{1}{\sqrt{17}}\left(\frac{-3+17}{2}\right).$$

Thus $c = (1/\sqrt{17})\,\alpha$ and $d = (-1/\sqrt{17})\,\beta$. This gives

$$a_n = \frac{1}{\sqrt{17}}\left(\alpha^{n+1} - \beta^{n+1}\right),$$

i.e.

$$a_n = \frac{(3+\sqrt{17})^{n+1} - (3-\sqrt{17})^{n+1}}{2^{n+1}\cdot\sqrt{17}}$$

Check that $a_0 = 1$ and $a_1 = 3$. Use $\alpha^2 - \beta^2 = (\alpha-\beta)(\alpha+\beta)$ for checking $a_1 = 3$.

3. (10 points) Give a bijective proof of the equality

$$3^n - 2^n = \sum_{k=1}^{n} \binom{n}{k} 2^{n-k}$$

The left side counts the number of [3]-strings of length n which contain 3, for the number of [3]-strings which do not contain 3 is 2^n.

Here is another way of counting such strings. Let $k \in \{1, 2, \ldots, n\}$. The number of [3]-strings of length n with exactly k 3's in them is $\binom{n}{k} \cdot 2^{n-k}$, for there are $\binom{n}{k}$ ways of picking places to put the k 3's in, and for the remaining n-k places we can slot in 1 or 2 in any way we choose to. It follows that the number of [3]-strings of length n which have at least one 3 in them is

$$\sum_{k=1}^{n} \binom{n}{k} 2^{n-k}.$$

This gives the result.

4. (10 points) Simone has 12 mint condition, limited edition collector cards of famous mathematical theorems (each card has a different theorem on it). She wants to display them in 4 distinct display cases. However, she can't decide how to distribute them, so she chooses a uniformly random assignment of cards, out of all the ways of distributing the cards so that each display gets at least one card.

(a) How many different ways can Simone distribute the cards?

(b) What is the probability that every display gets 3 cards?

(c) What is the probability that one display gets 9 cards?

a) A way of distributing is the same as a map $f : [12] \longrightarrow [4]$ (why?). The requirement that every display has a card is the requirement that f is surjective. The # of surjective maps from $[12]$ to $[4]$ is

$$\sum_{k=0}^{4} \binom{4}{k} (-1)^k (4-k)^{12}.$$

(b) There are $12!/(3!\,3!\,3!\,3!)$ ways of putting 3 cards in each display. (One way to see this: Pick 3 cards from 12 to put into the first display; 3 from the remaining 9 to put in the second; 3 from the remaining 6 to put in the third; and put the remaining 3 in the last display. Check that $\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3} = 12!/(3!\,3!\,3!\,3!)$.)

The required probability $= \dfrac{12!}{3!\,3!\,3!\,3!} \cdot \dfrac{1}{\sum_{k=0}^{4} \binom{4}{k}(4-k)^{12}}$

(c) There are four ways to choose the display to put the 9 cards in, $\binom{12}{9}$ ways of picking 9 cards from 12, and $3! = 6$ ways of placing the remaining 3 cards in the 3 displays that remain. This gives us $4 \cdot \binom{12}{9} \cdot 6$ ways of distributing the cards so that one display has 9 cards and the four displays have at least one card each.

The required probability is

$$\frac{24\binom{12}{9}}{\sum_{k=0}^{4} \binom{4}{k}(-1)^k (4-k)^{12}}$$

Don't have to simplify

5. (10 points) For the folowing graph, determine:

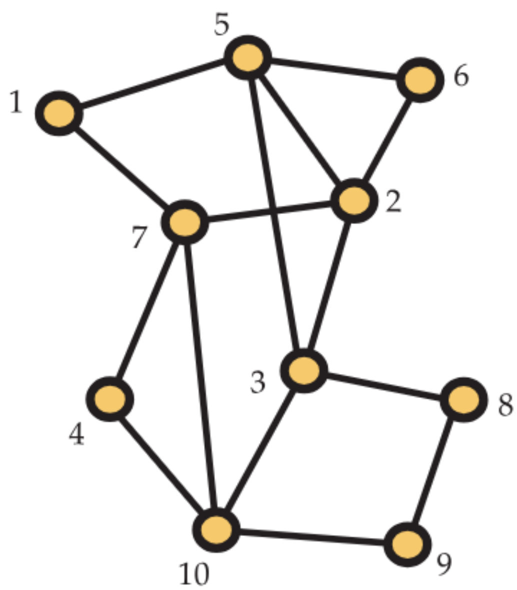
(a) if it is Eulerian;

(b) if it is Hamiltonian;

(c) if it is planar;

(d) its chromatic number;

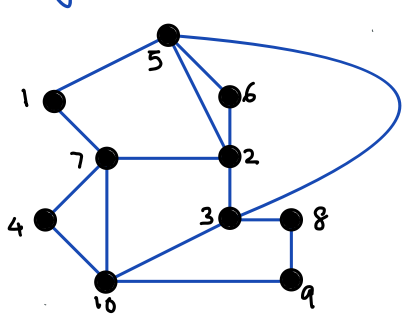
(e) its maximum clique size.



(a) deg (1) = 2, deg (2) = 4, deg (3) = 4, deg (4) = 2, deg (5) = 4, deg (6) = 2, deg (7) = 4, deg (8) = 2, deg (9) = 2, deg (10) = 4. Since all degrees are even and the graph is connected, it is eulerian.

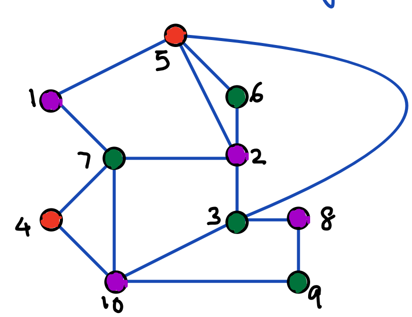
(b) The cycle (1, 7, 4, 10, 9, 8, 3, 2, 6, 5) is hamiltonian, so the graph is hamiltonian.

(c)



It is planar (see picture above)

(d)



The above colouring scheme shows that the chromatic # ≤ 3. On the other hand, the cycle (4, 7, 10) forces us to use three colours. So the chromatic number is 3.

(e) We did not do cliques in the course. Ignore.

6. (10 points) Show that there is no graph with at least 2 vertices where all the vertices have distinct degrees.

Let $G = (V,E)$ be a graph s.t. $|V| \geq 2$. We have to show that at least two distinct vertices of G have the same degree.

If $\deg_G(x) = 0$ for every $x \in V$, we are done, since G has two or more vertices. If $\deg_G(x) \geq 1$ for some, then the connected component of G which has x as a vertex has at least two vertices, and if we show that this connected component has two distinct vertices, we are done. So, without loss of generality, we may assume that G is connected. (We are using the fact that if H is a connected component of G and x is a vertex of H, then $\deg_H(x) = \deg_G(x)$.)

Since G is connected, $\deg_G(x) \geq 1$ for every $x \in V$. Indeed, if $\deg_G(x) = 0$ for some $x \in V$, then x is an isolated vertex, and since G is connected, $V = \{x\}$, contradicting the hypothesis that $|V| \geq 2$.

Let $n = |V|$. By definition of degree, $\deg_G(x) \leq n-1$ for every $x \in V$. Moreover, $n \geq 2$ and G is connected, and hence $\deg_G(x) \geq 1$ $\forall x \in V$. Thus

$$1 \leq \deg_G(x) \leq n-1 \qquad \forall x \in V.$$

We have n vertices and $n-1$ possible degrees. By the Pigeonhole Principle, at least two distinct vertices have the same degree.

q.e.d.

7. (10 points) A set of integers A contains an *arithmetic progression of length* 3 if there exist integers a and $r \neq 0$ such that A contains $\{a, a+r, a+2r\}$. Suppose Bob is given a set of n integers, and is able to do elementary arithmetic operations (add, multiply, subtract, divide) with numbers in the set. Approximately how many arithmetic operations does Bob need to perform in order to determine if his set contains an arithmetic progression of length 3?

We haven't discussed such problems and they won't appear in this terms exams

8. (10 points) Suppose we choose integers a, b, c, d from the set of non-negative integer solutions to

$$a + b + c + d = 100$$

uniformly at random. What is the expected value of $a + b$?

It is clear, by symmetry, that

$$E(a) = E(b) = E(c) = E(d) \quad \text{———} \quad (*) .$$

Thus

$$\begin{aligned} 4E(a) &= E(a) + E(b) + E(c) + E(d) \quad (\text{by } (*)) \\ &= E(a+b+c+d) \\ &= E(100) = 100. \end{aligned}$$

This gives

$$E(a) = 25.$$

Now $(*)$ also gives $E(a+b) = 2E(a)$.

Hence

$$E(a+b) = 50 \longleftarrow \text{Answer.}$$

Note: Warren Zhu (a student) gave this very elegant solution!

9. (10 points) Show that the function $f(n) = 3^n - n2^n$ satisfies the recurrence $f(n) = 7f(n-1) - 16f(n-2) + 12f(n-3)$ with initial conditions $f(0) = f(1) = f(2) = 1$.

This is straightforward. The characteristic polynomial is

$$x^3 - 7x^2 + 16x - 12 = (x-3)(x-2)^2.$$

The general solution is

$$f(n) = \alpha \cdot 3^n + \beta \cdot 2^n + \gamma n \cdot 2^n, \quad n \in \mathbb{N}_0.$$

where α, β, and γ are constants. If we set $\alpha = 1$, $\beta = 0$, and $\gamma = -1$ we see very easily that $f(0) = f(1) = f(2) = 1$.

Work out the details.