MAT344 FALL 2022
MIDTERM PROJECT

Due date: Oct 24, 2022 (on Crowdmark by midnight)

In what follows $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ denote the set of positive integers, the set of integers, and the set of real numbers respectively. $\mathbf{N}_0$ denotes the set of non-negative integers.

1. Let $1 \le p \le k \le n$ with $p, k, n \in \mathbf{N}$. Give a combinatorial proof of the following identity:

$$\binom{n}{k} = \sum_{i=p}^{n-k+p} \binom{i-1}{p-1}\binom{n-i}{k-p}.$$

Solution: The left side is the number of ways of choosing k elements from the set $[n]$.

Suppose we have k elements $x_1 < x_2 < \ldots < x_k$ in $[n]$ such that $x_p = i$. Then $\{x_1, \ldots, x_{p-1}\} \subset [i-1]$ and $\{x_{p+1}, x_{p+2}, \ldots, x_k\} \subset \{i+1, i+2, \ldots, n\}$. Looking at the sizes of the sets $\{x_1, \ldots, x_{p-1}\}$, $[i-1]$, $\{x_{p+1}, x_{p+2}, \ldots, x_k\}$, and $\{i+1, i+2, \ldots, n\}$, we see that this is possible if and only if $p-1 \le i-1$ and $k-p \le n-i$. These two conditions combine to give $p \le i \le n-k+p$. In other words if we have k elements $x_1 < x_2 < \ldots < x_k$ in $[n]$, then i is a possible value of x_p if and only if $p \le i \le n-k+p$.

Choosing k elements $x_1 < x_2 < \ldots < x_k$ from $[n]$ so that that $x_p = i$ is the same as choosing $p-1$ elements from $[i-1]$ and $k-p$ elements from $\{i+1, i+1, \ldots, n\}$. There are clearly $\binom{i-1}{p-1}\binom{n-i}{k-p}$ ways of doing this. Summing over i in the range $p \le i \le n-k+p$, we get all ways of choosing k elements in $[n]$. But this sum is the right side. Hence we are done. $\square$

2. Let $n \in \mathbf{N}$. Give a lattice path (diagonal or usual) proof of the identity

$$\binom{2n}{n} = \binom{2n-1}{n} + \sum_{k=1}^{n} \frac{1}{k}\binom{2k-2}{k-1}\binom{2n-2k}{n-k}.$$

Solution: The left side is $|S|$ where S is the set of diagonal lattice paths from (0, 0) to ($2n$, 0). We have two disjoint subsets of S, namely A consisting of those paths in S which pass through (1, 1) and B consisting of those which pass through (1, -1). Thus $|S| = |A| + |B|$.

Now elements of A can be identified with diagonal lattice paths from $(1,1)$ to ($2n$, 0) and elements of B with diagonal lattice paths from (1, -1) to $(2n, 0)$. From standard formulas, we see that $|B| = \binom{2n-1}{n}$.

In view of the above, we have to show that $|A| = \sum_{k=1}^{n} \frac{1}{k}\binom{2k-2}{k-1}\binom{2n-2k}{n-k}$. Let σ be a diagonal lattice path from (1, 1) to ($2n$, 0). Let k be smallest integer such that $(2k, 0)$ is on σ, i.e., ($2k$, 0) is the first hit of σ on the x-axis. Then ($2k-1$, 1) is necessarily on σ, and σ does not dip below $y = 1$ from (1, 1) to ($2k-1$, 1). This is the same as a Catalan path from $(0,0)$ to $(2k-2, 0)$ translated to $(1,1)$, and there are C_{k-1} such paths, where C_{k-1} is the $(k-1)^{\text{th}}$ Catalan number. Once σ hits $(2k, 0)$ it can take any diagonal lattice path from $(2k, 0)$ to reach $(2n, 0)$ and the number of such paths is the same as the number of diagonal lattice paths from $(0, 0)$ to $(2(n-k), 0)$ and this number is $\binom{2n-2k}{n-k}$. Thus $|A| = \sum_{k=1}^{n} C_{k-1}\binom{2n-2k}{n-k}$. Since

$$C_{k-1} = \frac{1}{k}\binom{2k-2}{k-1},$$

we are done. □

3. Let $0 \le m \le n$. Give a combinatorial proof of the identity

$$3^{n-m}\binom{n}{m} = \sum_{r=m}^{n} 2^{n-r}\binom{n}{r}\binom{r}{m}.$$

Solution: Here are two solutions to the problem.

I. Consider the number of ways of forming a committee from a pool of n people, a sub-committee from the committee, and sub-committee of the sub-committee (a *sub-subcommittee* for short), with the requirement that the sub-subcommittee has exactly m members.

There are (at least) two methods of forming the committee. One method is: We select the m members of the sub-subcommittee first, and then decide the roles of the remaining $n-m$ people in the pool. There are $\binom{n}{m}$ ways of selecting the sub-committee. The possible roles for the remaining $n-m$ people are three: sub-committee member but not a sub-subcommittee member, committee member but not a subcommittee member, and non-member. There are 3^{n-m} possibilities. Thus the number of ways of forming the committee so that the stated requirements are satisfied is $3^{n-m}\binom{n}{m}$, which is the left side.

One could also form the committee by picking the sub-committee, and then picking the sub-subcommittee and then picking the remaining members of the committee. Before that, we need to pick the size r of the subcommittee. Clearly $m \le r \le n$. Having picked r, there are $\binom{n}{r}$ ways of forming a sub-committee of size r, and having picked the sub-committee, there are $\binom{r}{m}$ ways of forming the sub-subcommittee. The $n-r$ persons not in the sub-committee have two roles: non-members, or members of the committee who are not in the subcommittee. There are 2^{n-r} ways of assigning these roles. Thus the number of ways of forming the committee so that the sub-committee has size r is $2^{n-r}\binom{n}{r}\binom{r}{m}$. Summing over r in the range $m \le r \le n$ we see that the number of ways of forming the committee with all the requirements in place is $\sum_{r=m}^{n} 2^{n-r}\binom{n}{r}\binom{r}{m}$. This exactly the right side.[1]

II. Consider the set S of strings of length n from the set $X = \{0, 1, 2, 3\}$ such that there are exactly m zeroes in the string. There are $\binom{n}{m}$ ways of picking the spots where the 0's occur. The remaining spots in the string can be any element from $\{1, 2, 3\}$. Thus $|S| = 3^{n-m}\binom{n}{m}$ which is the left side.

We can also count the number of elements in S is a different way. Let r be an integer between m and n. Let S_r be subset of S consisting of strings such that the total number of 0's and 1's in the string is r. It is clear that $|S| = \sum_{r=m}^{n} |S_r|$. An element of S_r is a choice of r places in the string (amongst n) to place the 0's and

[1] What if we picked the committee first, then picked a sub-committee from the pool of selected committee members, and finally picked the sub-subcommittee? What answer will you get?.

1's, followed by a choice of m places from these r places to place the $0'$s, followed by an assignment of 2 or 3 in the remaining $n-r$ places. There are $\binom{n}{r}\binom{r}{m}2^{n-r}$ ways of doing this, whence $|S_r| = 2^{n-r}\binom{n}{r}\binom{r}{m}$. Since $|S| = \sum_{r=m}^{n} |S_r|$, it follows that $|S|$ is also the right side of asserted identity. □

Remark. The two solutions are really the same.

4. Let $G = (V, E)$ be a graph. An **eulerian trail** in G is a walk such that every edge is traversed exactly once. It differs from an eulerian circuit in that that the starting and the ending vertex in our walk need not be the same (an eulerian circuit is always an eulerian trail, but not every eulerian trail is an eulerian circuit). Prove that if G is connected and has at most two vertices of odd order, then it has an eulerian trail.

Solution: Suppose G is connected and has at most two vertices of odd degree. The number of vertices of odd degree is even (from a result proved in class). So the only possibilities are that G has no vertices of odd degree or it has two vertices of odd degree. In the first case G has an eulerian walk and so we are done. In the second case, suppose u and v are the two vertices with odd degree. Form a new graph $G' = (V', E')$ by adding an extra vertex x to G and an adding an edges from u to x and v to x. Thus $V' = V \cup \{x\}$ and $E' = E \cup \{ux, vx\}$. Then G' is connected and every one of its vertices has even degree. It follows that G' has an eulerian walk starting and ending at x. Remove the edges xu and vx from this walk, and we have a walk in G from u to v which visits every edge in G. □

5. Let $n \in \mathbf{N}$ and $X = [n] = \{1, 2, \dots, n\}$. We say that a permutation π of X (by which we mean a permutation of length n of X) has a *descent* at position k if $\pi(k) > \pi(k+1)$.

(a) Fix k such that $1 \leq k \leq n$. How many permutations of X are such that they have have exactly one descent and that descent is at k?

(b) How many permutations of X have at most one descent?

Solution: A permutation $\pi = \pi_1\pi_2\dots\pi_n$ has descent at exactly k if and only if it simulatenously satisfies

(i) $\pi_1 < \pi_2 < \cdots < \pi_k$,
(ii) $\pi_k > \pi_{k+1}$, and
(iii) $\pi_{k+1} < \pi_{k+2} < \cdots < \pi_n$.

Choose a subset S of $[n]$ of size k arrange the elements of S as $\pi_1 < \pi_2 < \cdots < \pi_k$. Arrange the remaining elements in $[n]$ as $\pi_{k+1} < \pi_{k+2} < \cdots < \pi_n$. The only way that $\pi_k < \pi_{k+1}$ is if $S = \{1, 2, \dots, k\}$, for then $\pi_{k+1} = k+1$. For every other possibility for S, $\pi_k > \pi_{k+1}$. Thus the number of permutations with descent at exactly k is $\binom{n}{k} - 1$. This answers part (a).

Now for part (b). There is only one permutation with no descents, and that is $\pi = 123\dots n$. The number of permutations with at most one descent is the sum of the number of permutations with no descents, the number with exactly one descent at position 1, exactly one descent at position 2, at position 3, ..., at position $n-1$.

This sum is

$$1 + \left[\binom{n}{1} - 1 + \binom{n}{2} - 1 + \dots \binom{n}{n-1} - 1 \right] = 1 + \sum_{k=1}^{n-1} \binom{n}{k} - (n-1)$$
$$= 2 + \sum_{k=1}^{n-1} \binom{n}{k} - n$$
$$= \sum_{k=0}^{n} \binom{n}{k} - n$$
$$= 2^n - n.$$

This proves part (b). It is not necessary to do the above simplification to get full credit.

There is an equivalent way of doing part (b). We are really counting all the subsets of $[n]$ except the subsets $\{1\}$, $\{1, 2\}$, $\dots$, $\{1, 2, \dots, n\}$. The empty set accounts for the only permutation with no descent. Since there are 2^n subsets of $[n]$, we once again get $2^n - n$ as the answer for (b). □

6. Let $k \in \mathbf{N}$ and let X be a set with $|X| = k + 1$.

(a) Show that the number of X-strings of length n with no two consecutive repeated characters of the form xx in the string is $(k + 1)k^{n-1}$.

(b) Fix a character y in X and consider the set P_n of X-strings of length n with no two consecutive y's in them. Find a recurrence relation and initial conditions for the function $f(n)$ given by the formula $f(n) = |P_n|$. (You will have to give as many initial conditions as are needed to recursively determine $f(n)$ from earlier terms. For example, a recurrence relation of the form $g(n) = 7g(n - 1) + 4g(n - 2)$ requires two initial conditions.)

Solution: For part (a), there are $k + 1$ choices for the first spot in the string. Having picked a character for the first spot, we cannot use it for the second spot, and so we have only k choices for the second spot. Having picked a character for the second spot, we have only k choices for the third spot. Continuing this way we get that the number of strings with no two consecutive repeated characters is $(k + 1)k^{n-1}$.

Now for (b). It is clear that $f(1) = k + 1$, because no X-string of length 1 can possibly have two consecutive y's occurring in it. Next, there is only one X-string of length 2 which has two consecutive y's in it, namely the string yy. Since there are $(k + 1)^2$ X-strings of length 2, it follows that $f(2) = (k + 1)^2 - 1$.

Now suppose $n \geq 3$. Let $\mathbf{x} = x_1 x_2 \dots x_n$ be an X-string of length n. There are two possibilities,:

- $x_n \neq y$,and
- $x_n = y$.

In the first case, there are k choices for x_n. Moreover, in this case, $\mathbf{x}$ has no consecutive y's if and only if the string $x_1 x_2 \dots x_{n-1}$ has no consecutive y's. There are $f(n - 1)$ choices of these, and k choices for x_n. Thus the number of strings $\mathbf{x} = x_1 x_2 \dots x_n$ of length n, with $x_n \neq y$ and with no two consecutive y's is $kf(n-1)$.

If $x_n = y$, then $\mathbf{x}$ has no consecutive y's if and only if $x_{n-1} \neq y$ and the string $x_1 x_2 \dots x_{n-2}$ has no consecutive y's. There are k choices for x_{n-1} and $f(n - 2)$

choices for $x_1x_2 \dots x_{n-2}$, and hence the number of choices for $\mathbf{x}$ with $x_n = y$ and without consecutive y's is $kf(n-2)$. Thus $f(n)$ satisfies the recurrence relation

$$f(n) = k\Big(f(n-1) + f(n-2)\Big), \qquad n \geq 3,$$

with the initial conditions $f(1) = k + 1$, and $f(2) = (k+1)^2 - 1$.