Feb 1, 2018 Lecture 9

Here are three theorems stated without proof in the book.

Theorem 1 (Dirac, 1952) : A graph with n vertices, $n>2$, has a Hamilton circuit if the degree of each vertex is at least $n/2$

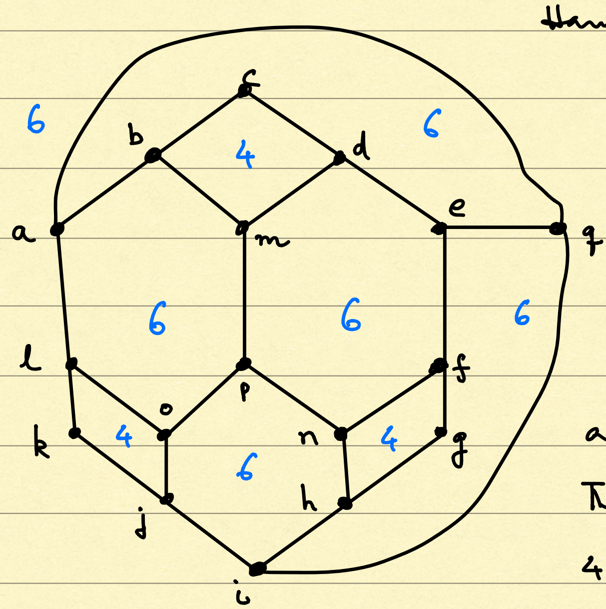
Theorem 2 (Chvatal, 1972) : Let $G=(V,E)$ be a graph and suppose $V=\{x_1,\dots,x_n\}$ with $\deg x_i \le \deg x_{i+1}$ (Note : we can always index the elements of V so that this condition is satisfied.) If for each $k \le \frac{n}{2}$ either $\deg(x_k) > k$ or $\deg(x_{n-k}) \ge n-k$, then G has a Hamilton circuit.

Theorem 3 (Grinberg, 1968) : Suppose a planar graph G has a Hamilton circuit H. Let G be drawn with any planar depiction, and let r_i denote the number of regions inside the Hamilton circuit bounded by i edges in this depiction. Let r_i' be the number of regions outside the circuit bounded by i edges. Then the numbers r_i and r_i' satisfy the eqn.

$$\sum_i (i-2)(r_i - r_i') = 0 \qquad (*)$$

The example below is an application of Theorem 3.

Example: Consider the graph below. We claim it has no Hamilton circuit.



The numbers inside regions is the number of edges bounding the region (the "outside" also has a number, namely 6). The numbers are either 4 or 6.

Thus we only have r_4, r_4', r_6 and r_6'.

There are 3 regions with 4 edges and 6 regions with 6 edges. It follows that

$$r_4 + r_4' = 3 \quad \text{and} \quad r_6 + r_6' = 6.$$

According to (*)

$$2(r_4 - r_4') + 4(r_6 - r_6') = 0.$$

It turns out that the three equations do not have a simultaneous integer non-negative solution. (See textbook, pp. 61–62).

Definition: A tournament is a directed graph obtained from a complete (undirected) graph by giving a direction to each edge.

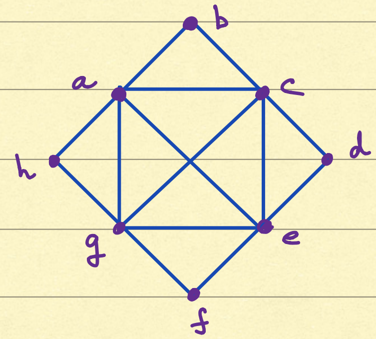
Theorem: Every tournament has a Hamilton path.

Proof: Easy. See textbook.

Section 2.3 Graph Colouring

Let G be a graph. A colouring of G assigns colours to vertices of G so that no two adjacent vertices have the same colour. The minimal number of colours required to colour G is called the chromatic number of G and is denoted $\chi(G)$.

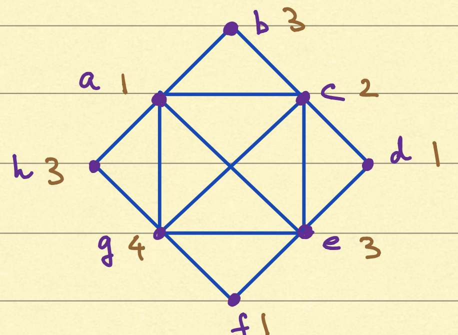
Example



Let us use 1, 2, 3, 4 etc as "names" of our colours. Suppose we colour a as 1.

a $\longleftrightarrow$ 1

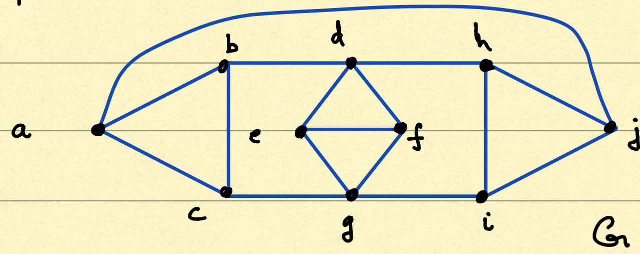
Then c, g, e have to be assigned a colour different from 1, and from each other (since c, g, e are pairwise adjacent). So suppose we use 2 for c, 3 for e and 4 for g. One therefore needs at least 4 colours to colour this graph, which we call G.



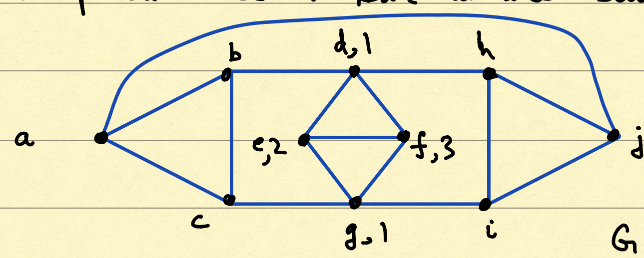
So $\chi(G) \geq 4$.

On the other hand 4 colours are enough to colour G as the picture shows. Thus $\boxed{\chi(G) = 4.}$

Example

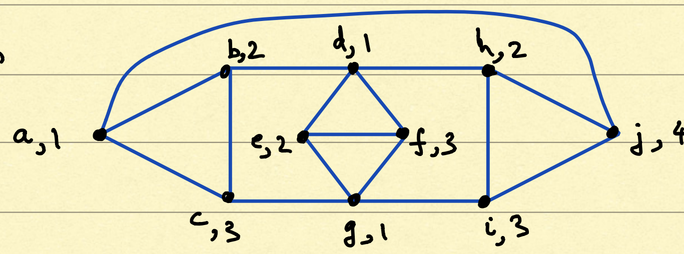


Because of the triangles we need at least 3 colours. Suppose we try to colour it with 3 colours. Colour d 1, e 2, and f 3. Then g is forced to be 1. Here is the situation.



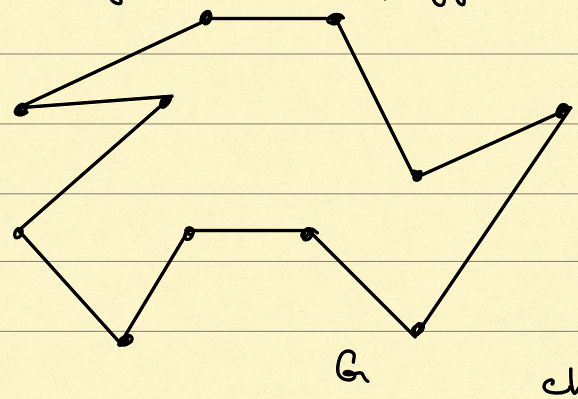
Vertices b and c cannot be coloured 1. And they cannot be coloured by the same colour. So assume b is coloured 2 and c is coloured 3 (it does not matter by symmetry). This forces a to have colour 1. Similar argument using h and i (which have to have colours from 2 and 3) we see that a and j have to have colour 1. But this is not possible since a & j are adjacent. So 3 colours are not sufficient to colour G. This means $\chi(G) \geq 4$. The following picture shows that 4 colours suffice. Thus

$$\boxed{\chi(G) = 4.}$$



2.4. Colouring Theorems

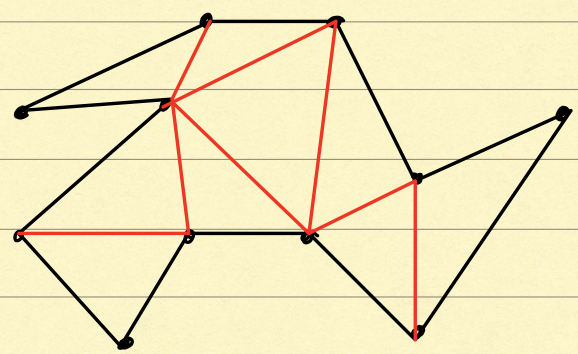
Triangulation of a polygon.



Here is a polygon.

Need not be convex.

A triangulation of a polygon is the process of adding straight line chords between pairs of vertices so that all interior regions are bounded by triangles.



← Here is a triangulation of G.

Theorem 1: The vertices in a triangulation of a polygon can be 3-coloured.

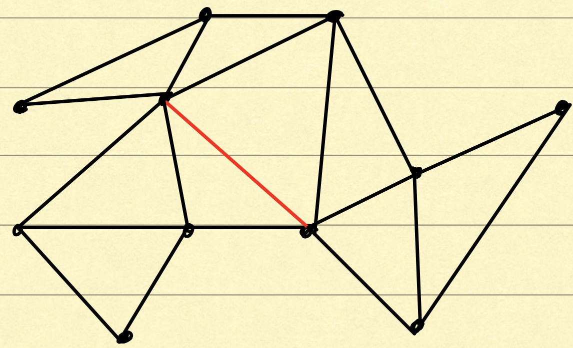
Proof: Let G be the triangulated polygon.

Let $n =$ # of edges of the polygon.

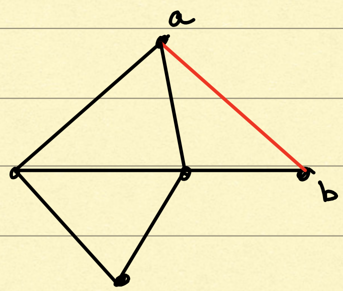
For $n = 3$, give each corner a different colour.

Assume the theorem is true for any triangulated polygon with fewer than n vertices.

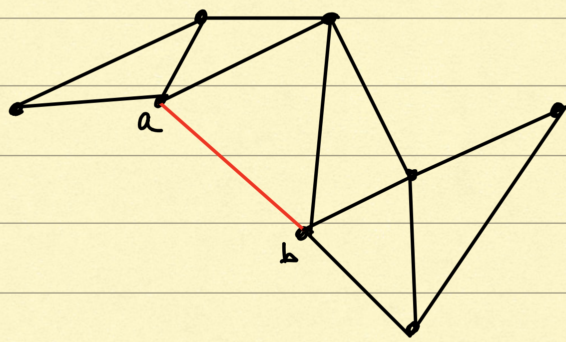
Pick a chord. This splits the polygon into smaller polygons. For example, consider the chord coloured red below:



This splits the graph into two smaller ones G_1 and G_2



G_1



G_2

We can find a chord because $n \geq 4$.
By induction, since G_1 and G_2 have fewer than n edges, each of them can be 3-coloured. Choose the same colours for the corresponding vertices along the chords (a,b) in G_1 and G_2. Putting them back together we see that G can be 3-coloured.

q.e.d.

The Art Gallery Problem: The walls of an art gallery are assumed to form a polygon. What is the smallest number of guards needed to watch paintings along the n walls of a gallery? A guard at the corner of a wall is assumed to be able to see the two walls meeting in the corner. The guard can also see walls in direct line of sight.

In what follows, for a real number x, $[x]$ is the largest integer $\leq x$.

Corollary (to Thm 1) (Fisk, 1978): The Art Gallery problem with n walls requires at most $[\frac{n}{3}]$ guards.

Proof: Make a triangulation from the walls. Colour the vertices in three colours, say red, blue, and green. Station a guard at those corners of the original wall which are coloured "red". Check that such a stationing covers all walls. The polygonal wall has n corners. Some colour is used at $[n/3]$ or fewer corners.

q.e.d.

Here is an important theorem.

Theorem: Every planar graph can be 5-coloured.

Proof:

Enough to consider only connected graphs, since we can then 5-colour each connected component by using the same five colours for each component.

A 1-vertex graph can be trivially 5-coloured.

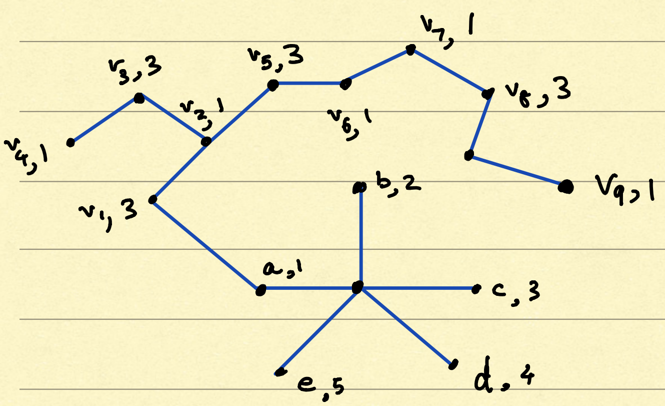
Suppose all connected planar graphs with $n-1$ vertices $(n \geq 2)$ can be 5-coloured. We now show that a connected planar graph with n vertices can be 5-coloured. By problem 18(a) in Section 1.4 (this is part of HW 2), G has a vertex x with $\deg(x) \leq 5$. Delete x from G

and all edges incident at x, to obtain a graph with $n-1$ vertices, which by assumption can be 5-coloured.

Reconnect x. We have to give a colour to x so that G is 5-coloured. If $\deg(x) \leq 4$ there is no problem. Just assign x a colour different from its neighbours.

If x has degree 5, but two neighbours have the same colour, then the same approach works.

The only complicated case is when $\deg(x) = 5$, and the five adjacent vertices a, b, c, d, e have five distinct colours, say 1, 2, 3, 4, 5. Colour a as 1, b as 2, c as 3, d as 4, and e as 5.



Consider all paths starting from a which have only 1 and 3 as colours. Call the resulting sub-graph G'. If c does not lie in G' then change the colours of the vertices in G', switching 1 to 3 and 3 to 1. The result is still a colouring of G, and c remains coloured 3, but now a is also of colour 3. We can colour x by 1.

If on the other hand c is a vertex in G', i.e. there is a path from a to c consisting of only 1 and 3 (call this path P), then we do the following. Consider all paths starting at b whose vertices are coloured either 2 or 4. This creates a subgraph G''. The vertex d cannot be in G''. If it is, there is a path Q from b to d and this has to cross the path P from a to c, violating planarity.

So d is not a vertex in G''. So we can exchange the colours 2 and 4 in all vertices in G''. The resulting scheme is a colouring of G. Since d is not in G'', its colour is 4. Now b and d both have colour 4, and x can be coloured 2.

q.e.d.