Jan 30, 2018

Lecture 8

The last time we stated the following theorem of Euler, which he proved in 1731 in connection with the Konigsberg Bridge Problem.

Theorem (Euler): A multigraph has an Euler cycle if and only if it is connected and has all vertices of even degree.

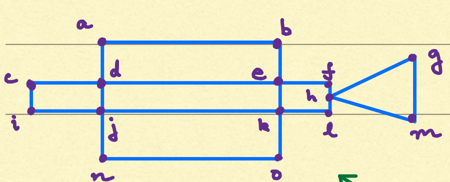
The proof uses (essentially) the method given in the example done in class.

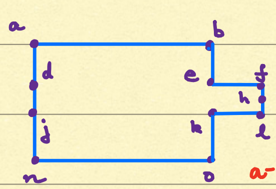
Proof of Euler's theorem:

We have already seen that if a multigraph has an Euler cycle then it is connected and all its vertices are of even degree. We now have to prove the converse.

Let G be a multigraph which is connected and whose vertices are all of even degree. We have to show that G must have an Euler cycle.

We use the method in the example. Recall the process.

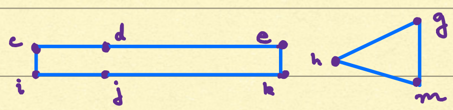




a-d-j-n-o-k-l-h-f-e-b-a

From the original graph find a cycle by starting at an arbitrary vertex and moving on edges incident on the vertex. Remove the cycle.

Get →



d-c-i-j-k-e-d

h-g-m-h

Incorporate $\underbrace{\text{these two}}$ into this to get a new cycle, namely

a-d-c-i-j-k-e-d-j-n-o-k-l-h-g-m-h-f-e-b-a

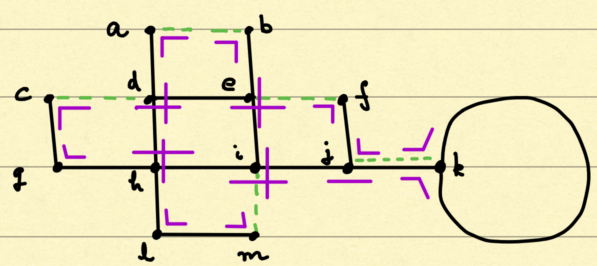
Start at an arbitrary vertex of your choice. Call it a. From a pick an edge and move to an adjacent vertex. Keep doing this, making sure no edge is repeated. Since there are only a finite number of edges, the process has to stop. Suppose the vertex we are visiting when the process stops is v. This means when we enter v during this last visit, all edges incident on v have been exhausted and used. We claim $v = a$. We will prove our claim by contradiction. Suppose $v \neq a$. Let m be the number of times v has been visited before the last visit (m could be zero). Then from earlier visits, $2m$ edges incident on v have been used, m to enter v, and m to leave v (remember $v \neq a$). The latest visit uses one more edge, and so $2m+1$ edges have been used. This means the degree of v is $2m+1$ which is odd. This contradicts the fact that the degree of v is even. This means $v = a$. Thus the process has given us a cycle since it begins and ends at a. It may or may not be an Euler cycle. If not do the following: Remove all the edges from this cycle from the graph and all vertices which become isolated (i.e., have no edges incident on them) after the edges in our cycle are removed. What remains is a possibly disconnected graph. In the new graph, all the vertices continue to have even degree, for the number of edges from the cycle incident on a vertex is even (every edge entering the vertex being matched by one leaving). Each component of the new graph has at least one vertex in common with the cycle removed, for the original graph is connected. So in each component, repeat the process, but starting from a vertex common to the cycle that has been removed. These new cycles can be incorporated into the original cycle by using exactly the same procedure we used in the example. We now have a larger cycle. Remove this larger cycle and repeat the process until all edges and vertices have been used. What we get is an Euler cycle.

q.e.d.

Question: Where in the proof of the existence of an Euler cycle have we used the two conditions given? Re-read the proof of the existence of an Euler cycle and identify the places where the hypothesis of even degree for vertices is used as well as the places where the connectedness of the graph is used.

Example (Routing street sweepers)

Here is a street map of streets to be swept by a street sweeper of a city. The solid lines represent blocks, and the circular arc a circular street.



Solid edges are streets. Dotted edges are introduced to make vertices have even degree. The dotted edges are called dead heading edges.

Want: A tour that sweeps each solid edge once. This means we want a tour of solid edges which is an Euler cycle. This is clearly not possible because many vertices have odd degree if we only worry about solid edges. So we are forced to introduce extra edges which the sweeper can traverse so that all vertices have even edges. These extra edges are called deadheading edges.

We now have a new problem. We would like to minimize the number of deadheading edges. (This could be to minimize labour, as well as time. For example, perhaps the streets have to be swept from 10 am to 11 am when parking on the blocks is forbidden.)

Suppose the dotted edges we have drawn represent a minimal number of deadheading edges. We then have a new problem. We would like to minimize the number of turns the sweeper makes to so that traffic does not get tied up (turns, right or left, do tie up traffic).

Start from a. Don't turn unless you have to. If we go clockwise, one ends up with the cycle

a - b - e - i - m - l - h - d - a.

Next start from c. Again do not turn unless you are forced to. Suppose we go clockwise (you can go counterclockwise too if you wish). The cycle you trace out is

c - d - e - f - j - k - k - j - i - h - g - c.

Now you can fuse the two cycles to get an Euler cycle by picking a common vertex. Suppose you pick d as the common vertex. Then one can make the following Euler cycle by fusing:

a - b - e - i - m - l - h - d - e - f - j - k - k - j - i - h - g - c - d - a

incorporating the 2nd cycle into the 1st

At d we have two turns. Otherwise we have minimal turns.

Another proof of the theorem on Euler cycles

Say G is a multigraph which is connected and each vertex has even degree. At each vertex pair off edges incident on the vertex. This can be done because the degree of each vertex is even. So each edge links to another edge and forms (eventually) a cycle. There are many cycles that get formed by this pairing. Fuse them together by using common vertices as in the example.

Definition: Let G be a multigraph. An Euler trail in G is a trail that contains all the edges in G and visits each vertex at least once.

Corollary: A multigraph has an Euler trail, but not an Euler cycle, if and only if it is connected and has exactly two vertices of odd degree.

Proof:

Suppose a multigraph G has an Euler trail but not an Euler cycle. If v is a vertex which is neither the starting point nor the ending point of a trail then every edge in the trail which enters v has a matching edge which leaves it, and since the trail is an Euler trail, every edge incident on v is either an edge through which one enters v or one through which one leaves v. This means v is has even degree.

Now consider the starting point of the Euler trail. Call it x. Any later visit to x must be followed by an exit, otherwise the trail is a cycle, and we are assuming G has no Euler cycle.

This means the degree of x is odd. Similarly if y the terminal point of the Euler trail, its degree is odd. And as we have seen, all vertices, other than x and y, have even degree.

Conversely, suppose G is connected and has exactly two vertices, say p and q, of odd degree. Introduce a new edge (p,q) to obtain new graph G'. Clearly G' is connected and all its vertices are of even degree. This means G' has an Euler cycle C. Remove the edge (p,q) from C. We get a trail all of whose edges are in G. This gives us an Euler trail in G.

q.e.d.

§2.2 Hamilton Circuits :

Definition : Let G be a graph. A Hamilton circuit in G is a circuit which visits each vertex in G exactly once. A Hamilton path is (similarly) a path in G which visits each vertex in G exactly once

Basic observations : Suppose C is a Hamilton circuit in a graph G. The following are easy to verify.

1. If v is a vertex in G of degree 2, then both the edges incident on v must be part of C.

2. C has no proper subcircuit, i.e., C has no subcircuit which is not C itself.

3. If v is a vertex and e, f are edges in C incident

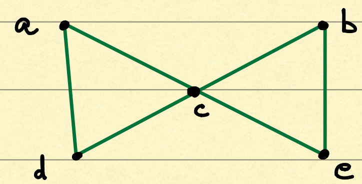
on v, then none of the other edges of G incident on v occurs in C.

In the textbook, these observations have been called "rules". Please see Rules 1, 2, and 3 on p. 57 of the textbook.

Examples of non-existence of Hamilton circuits

In the following examples, we will show that the given graphs do not have Hamilton circuits.

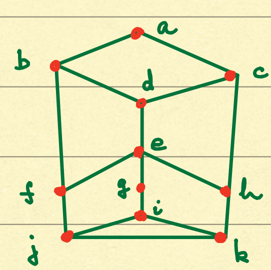
1.



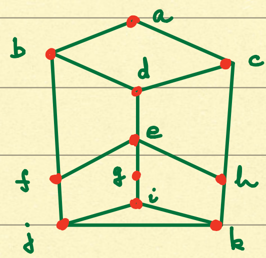
The degree 2 vertices are a, b, d, and e.

If the graph has a Hamilton circuit C, then by Rule 1 (i.e., observation 1) the edges (a,c), (a,d), (b,c), (b,e), (d,c), and (e,c) all are part of C. By Rule 3, only two edges incident on the vertex c can occur in the Hamilton circuit C. However, (a,c), (b,c), (d,c), (e,c), all occur in C. This is a contradiction of Rule 3. Hence the graph has no Hamilton circuit. (Rule 2 is also contradicted! Check!)

2. In the graph on the right, suppose we had a Hamilton circuit C.



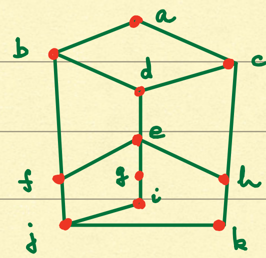
The vertices of degree 2 are a and g.



Since a and g have degree 2, all edges incident on them must be in C. (Rule 1)

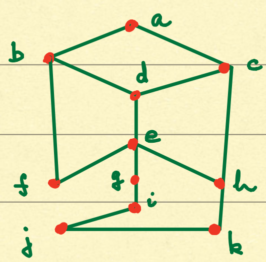
Let us concentrate on the vertex g. From what we just said, (e,g) and (i,g) both are part of C.

Since (i,g) lies in C, by Rule 3, exactly one of (i,j) or (i,k) lies in C, both cannot do so. Suppose (i,j) lies in C. Then (i,k) cannot. Let us delete (i,k)



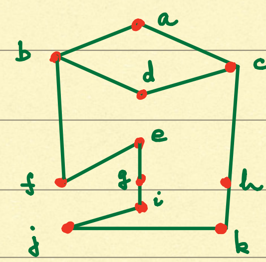
In the new picture, k has degree 2, and hence both (h,k) and (j,k) lie in C. (Rule 1)

Consider the vertex j. The edges (i,j) and (j,k) lie in C. By Rule 3, (f,j) cannot lie in C. Let us delete it.

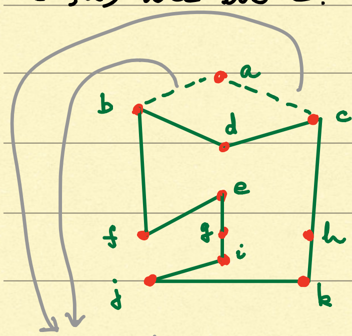


In the new picture, f has degree 2, so (b,f) and (g,f) must lie in C.

Look at the vertex e. We already have (e,g) in C. Now we have (f,e) in C. This means (by Rule 3) that (e,h) and (e,d) cannot be in C. Let us remove them. In the new graph d and h have degree 2, and so (b,d), (c,d), (c,h) and (h,k) must lie in C. If we look at vertex b, we see that (b,f) and (b,d) lie in C and so (b,a) cannot be in C. If we look at vertex c, we see that (c,d) and (c,h) lie in C,



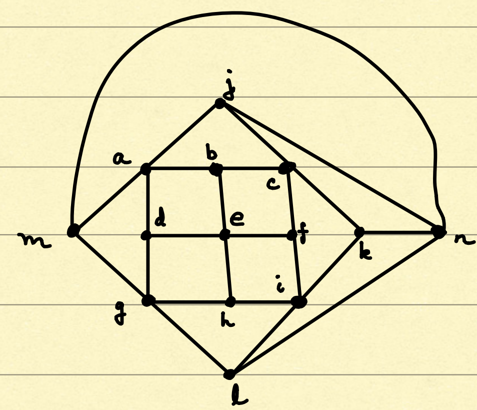
and this means (a, c) cannot lie in C. Thus neither (b, a) nor (c, a) are in C. On the other hand, the vertex a is of degree 2, which means both (b, a) and (c, a) are in C. This is a contradiction.



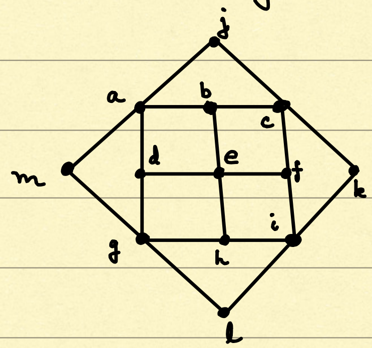
In C and not in C at the same time, giving a contradiction.

By symmetry, if (i, k) was in C, we would have reached the same conclusion. So the graph does not have a Hamilton circuit.

3. Consider the graph shown.



If we imagine the vertex n to lie above the vertex e in 3 dimensions, and (n, m), (n, j), (n, k), (n, l) to be straight edges in 3 dimensions, then we see that the graph has symmetry parallel to the x-axis as well as the y-axis.

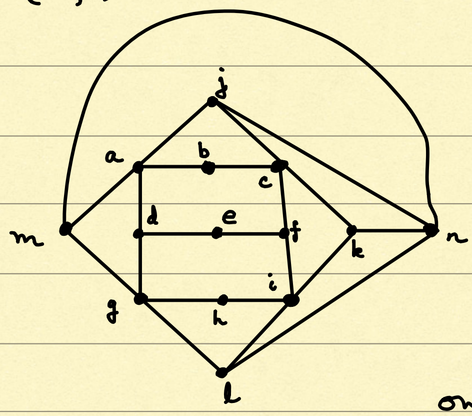


Imagine n to lie on the z-axis directly above e, and with edges connecting n to k, m, j and l. Then one can detect symmetry.

We claim that the above graph has no Hamilton circuit. To see this consider the vertex e. Suppose the graph had a

Hamilton circuit C. Exactly two edges incident on e lie in C. Either the two edges are incident on e from opposite sides, or they form a 90° angle.

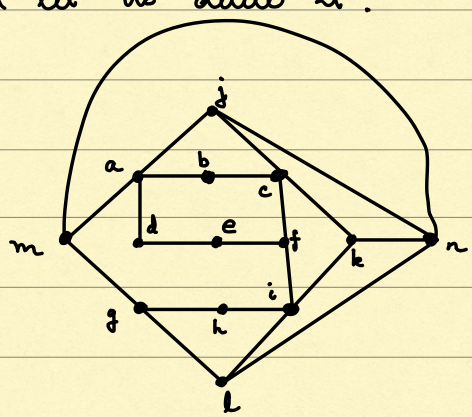
Case 1: Suppose the two edges are incident at e from opposite sides. The two edges could then either be (d,e) and (f,e) or they can be (b,e) and (h,e). By symmetry we can pick either pair. So suppose (d,e) and (f,e) lie in C. Then we can delete (b,e) and (h,e).



Now b and h have deg 2. So (b,a), (b,c), (h,g), and (h,i) are part of C.

Consider vertex d.

Since (d,e) is part of C, exactly one of (d,a) or (d,g) is in C. By symmetry, it does not matter which. For the sake of definiteness, suppose (d,a) is in C. Then (d,g) is not in C, and let us delete it.

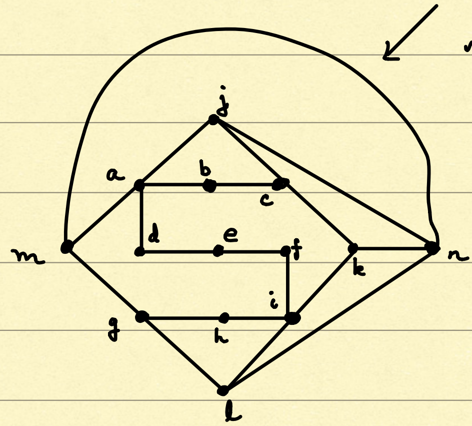


Thus, we have (a,d), (b,a), (b,c), (d,e), (e,f) in C. The edge (c,f) therefore cannot be in C, for then we would have a proper sub-circuit

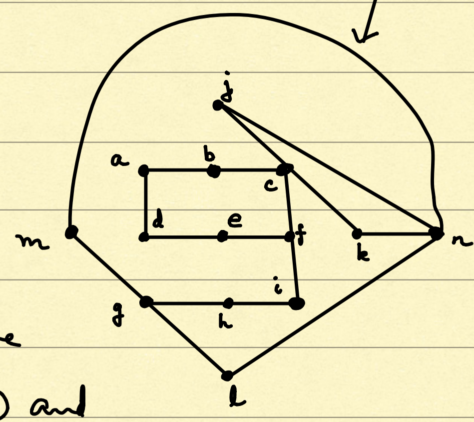
a-b-c-f-e-d-a

which violates Rule 2.

Let us remove (c,f) then. Clearly (a,m) and (a,j) cannot be in C, for (a,d) and (a,b) are in C.



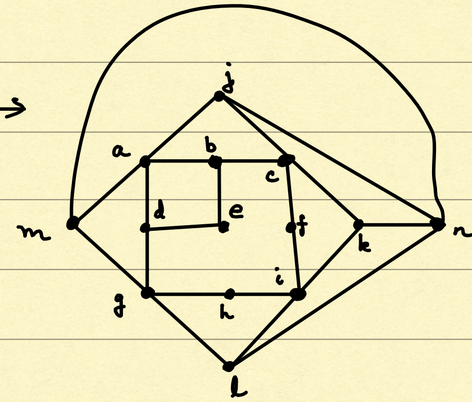
Similarly (i,l) and (i,k) cannot be in C. Let us remove these. We get



In this $\longrightarrow$ graph, j, m, l, k have degree 2 and so all edges incident on them are in C. This forces (j,n), (m,n), (k,n) and (l,n) to be in C. This violates Rule 3. So in this case we have no Hamilton circuit.

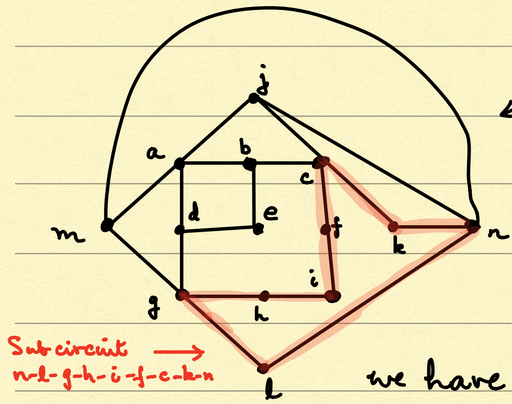
<u>Case 2</u>: The two edges in C incident at e meet at $90°$. By symmetry we may assume (b,e) and (d,e) are in C. So we have to delete (f,e) and (h,e).

We get this picture $\longrightarrow$



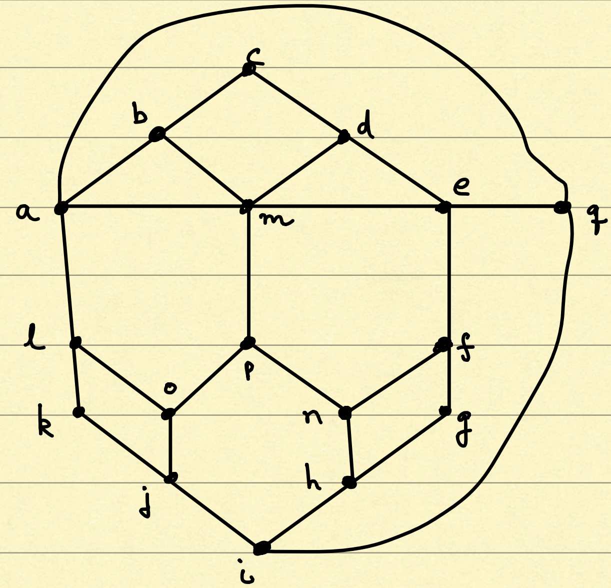
Now f and h have deg 2. So (f,i) and (h,i) must be in C. This forces us to delete the edges (i,k) and (i,l).

The resulting graph is below.



In this graph k and l have degree 2. This means (c,k), (k,n), (l,g) and (l,n) must be in C. We already have seen that (g,h), (h,i), (f,i), (c,f) are in C. Thus we have a proper subcircuit n-l-g-h-i-f-c-k-n. This violates Rule 1. Thus the graph has no Hamilton circuit.

4. Consider the graph below:



It turns out this does not have a Hamilton circuit. To do this problem read Theorems 1, 2, 3 on p 61 of the textbook. The above graph has no Hamilton circuit because of Theorem 3, i.e., Grinberg's theorem. Read the solution from the book.