Jan 25, 2018

Lecture 7

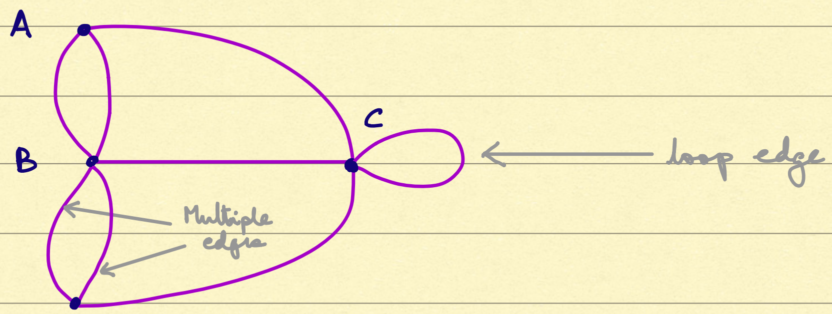
Chapter 2

Important concepts in this chapter are (definitions etc later):

- Multigraphs } ⟵ Section 2.1
- Euler cycles }
- Hamilton circuits and paths ⟵ Section 2.2

Section 2.1 (Multigraphs and Euler cycles)

Definition: A multigraph $G=(V,E)$ is a set of vertices V and a set of edges E between vertices in which several edges are allowed to join two vertices and loop edges, i.e. edges of the form (a,a) for $a \in V$ are also allowed.

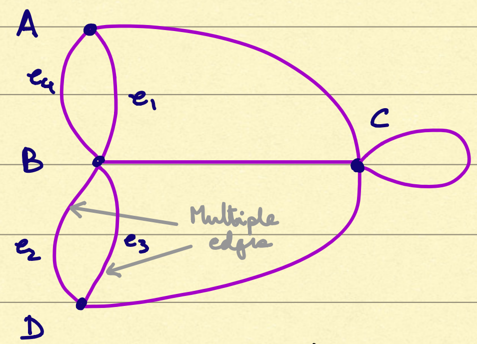


Example of a multigraph

Note: A usual graph is also a multigraph.

Trails and cycles in a multigraph

A **trail** in a multigraph is a sequence of vertices, not necessarily distinct, together with edges joining successive vertices, with no edge repeated in a trail.

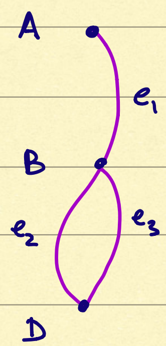


on the right

The A – B – D – B is a trail. Better to use edges to denote trails.

left right

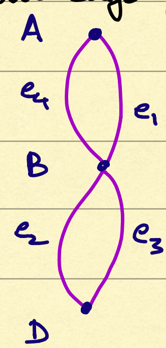
If e_1, e_2, e_3, e_4 are the edges in the picture, then $e_1 e_2 e_3$ is a trail.



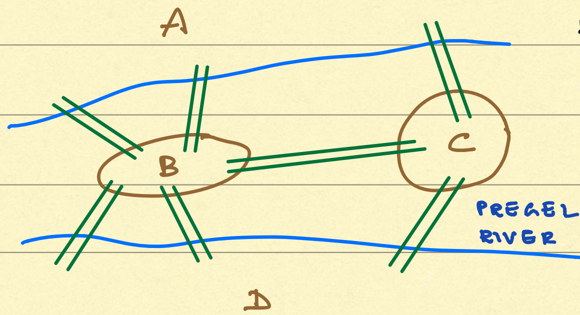
Trail. Note that the vertex B is visited twice. The trail is e_1 followed by e_2 followed by e_3. Can also have a different trail, namely e_1 followed by e_3 followed by e_2. Once again, you are reminded that edges cannot be repeated in a trail.

A **cycle** is a trail together with an edge joining the last vertex to the first.

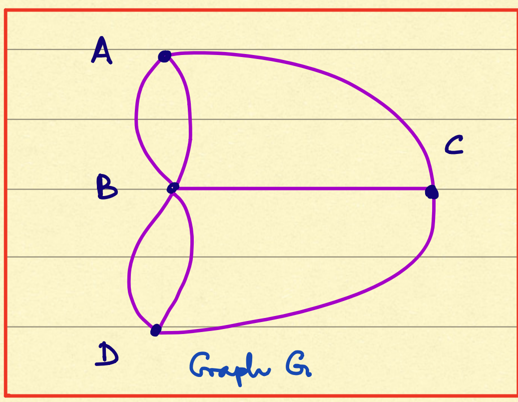
In this picture $e_1 e_2 e_3 e_4$ is a cycle, as is $e_1 e_4$, or $e_3 e_2$, or $e_1 e_3 e_2 e_4$, or $e_4 e_1$ or ...



The seven bridges of Konigsberg



Konigsberg (in Prussia) was located on Pregel River and had two islands (denoted B and C in the picture) with the banks A and D. There were seven bridges connecting A, B, C, and D, as shown in the picture above. The townspeople used to take walks across the bridges. As a recreational problem (i.e., for fun) they wanted to figure out a walk which crossed each bridge just once. They approached Leonhard Euler to solve the problem. This is the problem which gave birth to graph theory. Euler solved the problem and essentially invented graph theory with his solution.



Graph G

We can model the problem by the multigraph on the left, with the banks A, D and the islands B, C being depicted by vertices, and each bridge representing an edge (see Graph G above). The problem is to find a cycle that contains all the edges in a graph and visits each vertex at least once. Such cycles are called Euler cycles. Check that there is NO Euler cycle in G.

This leads us to a definition.

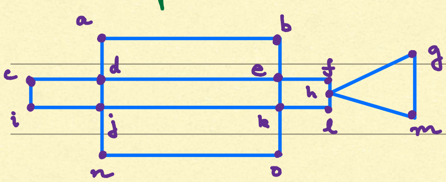
Definition: A cycle in a multigraph G is called an Euler cycle if it contains all the edges of G and visits each vertex at least once.

Note that if a multigraph G has an Euler cycle it must be connected since one has a path connecting all vertices. Moreover, if we have an Euler cycle C, then if an edge in the cycle visits vertex v, there is another leaving it. So the degree of each vertex is even.

In the graph for the Konigsberg bridge problem, the vertex c has degree 3, and so the graph cannot have an Euler cycle.

It turns out that the two conditions that a graph possesses if it has an Euler cycle are also sufficient conditions for the graph to have an Euler cycle.

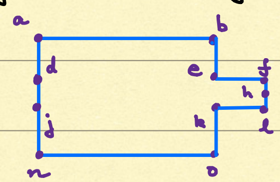
Example



Consider the graph on the left. It is connected and every vertex has even degree. How does one find an Euler cycle?

Start from a random vertex, say a. Trace a trail. Recall, this means vertices can be visited multiple times but edges cannot be repeated. Since every vertex has even degree, every vertex we enter can be left by another edge. In the end we have to stop at a. Here is a trail

a-d-j-n-o-k-l-h-f-e-b-a



Remove the above trail from the graph. Get the following disconnected graph.

Each connected piece has an Euler cycle. The first one has d-c-i-j-k-e-d and the second h-g-m-h. These two cycles can be inserted into the original cycle a-d-j-n-o-k-l-h-f-e-b-a at d and h to give the Euler cycle

$$\boxed{a-d-c-i-j-k-e-d-j-n-o-k-l-h-g-m-h-f-e-b-a.}$$

This is the method used by Euler in 1736 to prove:

Theorem (Euler): A multigraph has an Euler cycle if and only if it is connected and has all vertices of even degree.

Did not have time to present the proof today. Will prove it on Jan 30. If you are curious, here is the proof.

Proof of Euler's theorem:

We have already seen that if a multigraph has an Euler cycle then it is connected and all its vertices are of even degree. We now have to prove the converse.

Let G be a multigraph which is connected and whose vertices are all of even degree. We have to show that G must have an Euler cycle.

We use the method in the example.

Start at an abitrary vertex of your choice. Call it a. From a pick an edge and move to an adjacent vertex. Keep doing this, making sure no edge is repeated. Since there are only a finite number of edges, the process has to stop. Suppose the last vertex visited is v. This means when we enter v during this last visit all edges incident have been exhausted and used. We claim $v = a$. We will prove our claim by contradiction. Suppose $v \neq a$. Let m be the number of times v has been visited before the last visit (m could be zero). Then from earlier visits, 2m edges incident on v have been used, m to enter v, and m to leave v. The latest visit uses another edge, and so 2m+1 edges have been used. Since the degree of v is even, there has to be at least one more edge incident on v which has not been used (2m+1 is an odd number. We can use this edge to leave v, and so the process does not stop at v. This is a contradiction. This means $v = a$. Thus the process has given us a cycle since it begins and ends at a. It may or may not be an Euler cycle. If not do the following: Remove all the edges from this cycle from the graph and all vertices which become isolated (i.e., have no edges incident on them) after the edges in our cycle are removed. What remains is a possibly disconnected graph. In the new graph, all the vertices continue to have even degree, for the number of edges from the cycle incident on a vertex is even (every edge entering the vertex being matched by one leaving). Each component of the new graph has at least one vertex in common with the cycle removed, for the original graph is connected. So in each component, repeat the process, but starting from a vertex common to the cycle that has been removed. These new cycles can be incorporated into the original cycle by using exactly the same procedure we used in the example. We now have a larger cycle. Remove this larger cycle and repeat the process until all edges and vertices have been used.

What we get is an Euler cycle.

q.e.d.

Question: Where in the proof of the existence of an Euler cycle have we used the two conditions given? Re-read the proof of the existence of an Euler cycle and identify the places where the hypothesis of even degree for vertices is used as well as the places where the connectedness of the graph is used.