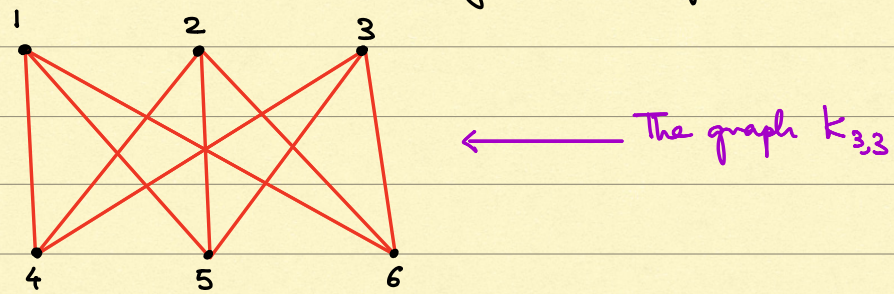
Jan 23, 2018

Lecture 6

Last time we used the circle-chord method to conclude that a certain graph was planar. Here is an example where the method allows us to conclude that a certain bipartite graph (called $K_{3,3}$) is non-planar.

Example: Consider the following bipartite graph:



It is denoted $K_{3,3}$ and it is a bipartite graph with three vertices in each member of the partition:

$$G = (V, E) = (V_1 \cup V_2, E)$$

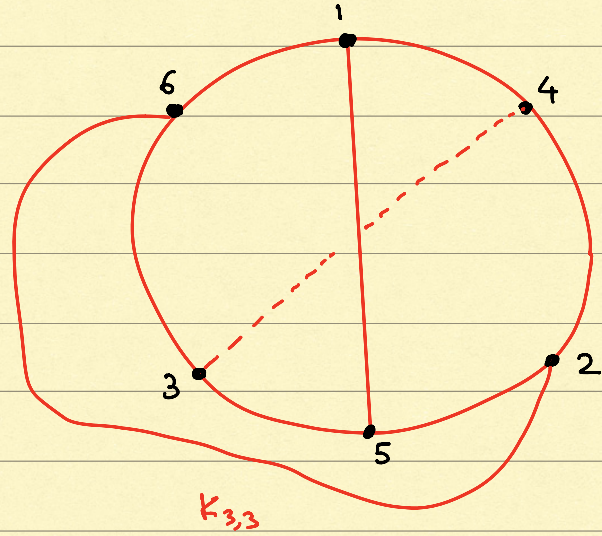
$$V = \{1, 2, 3, 4, 5, 6\}, \quad V_1 = \{1, 2, 3\}, \quad V_2 = \{4, 5, 6\}$$

with the added property that every vertex in V_1 is adjacent to all vertices in V_2 and vice-versa.

We will show $K_{3,3}$ is non-planar. Let us first look at a circuit containing all vertices. This is easy. In fact

$$C = 1-4-2-5-3-6-1$$

works. Again, there may be other possibilities for C.



$K_{3,3}$

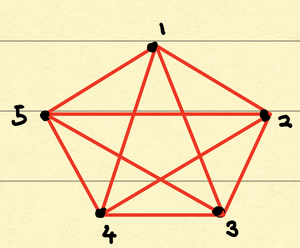
Draw $(1,5)$ "inside". Draw $(2,6)$ "outside". Whichever way you try to draw $(4,3)$ you will have to cross an older chord.

This shows $K_{3,3}$ is non-planar.

Exercise: Show using the circle-chord method that K_5 (the complete graph on 5 vertices) is non-planar.

Later in this or the next lecture, we will give another way of proving K_5 is non-planar using "Euler's formula" which is an important formula that we have to prove.

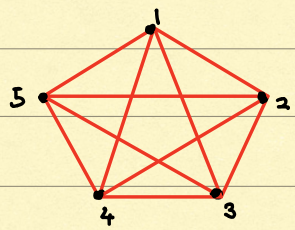
At the moment, we have two examples of non-planar graphs, namely, $K_{3,3}$ and K_5. In some sense, all non-planar graphs contain these two as subgraphs. This is not strictly accurate. For an accurate statement we need the notion of a configuration.



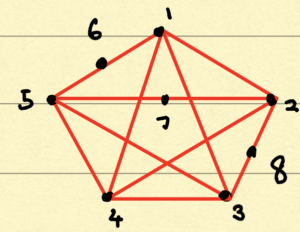
← Complete graph on 5 vertices.

K_5

Definition: Let G be a graph. A subgraph of G is called a $K_{3,3}$ configuration if it can be obtained from a $K_{3,3}$ by adding vertices in the middle of some edges. A K_5 configuration is defined similarly.



K_5



Note the extra vertices 6, 7, and 8

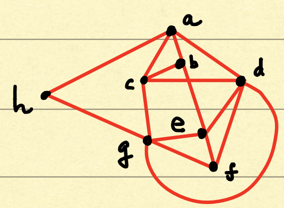
Configuration of K_5

The following important theorem is due to Kuratowski.

Theorem (Kuratowski): A graph is planar if and only if it does not contain a subgraph that is a K_5 or a $K_{3,3}$ configuration.

Note: It is not easy to find $K_{3,3}$ or K_5 configurations. However, most small non-planar graphs contain a $K_{3,3}$ configuration.

Example (Finding a $K_{3,3}$): Let us try to find a $K_{3,3}$ configuration in the figure below.

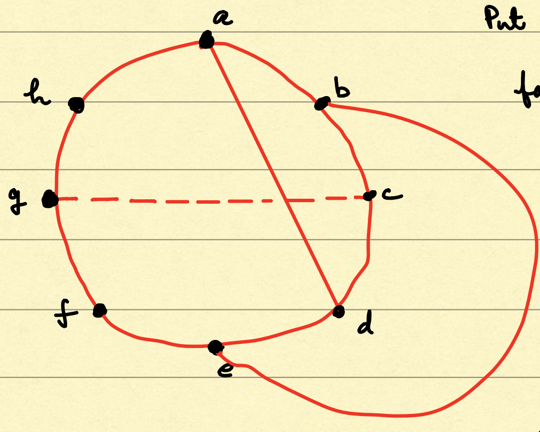


As usual, let us try to find a circuit that visits all vertices. It is easy to see that

$$C = a-b\ c-d-e-f-g$$

is such a circuit. Let us re-arrange vertices so that C is a "circle"

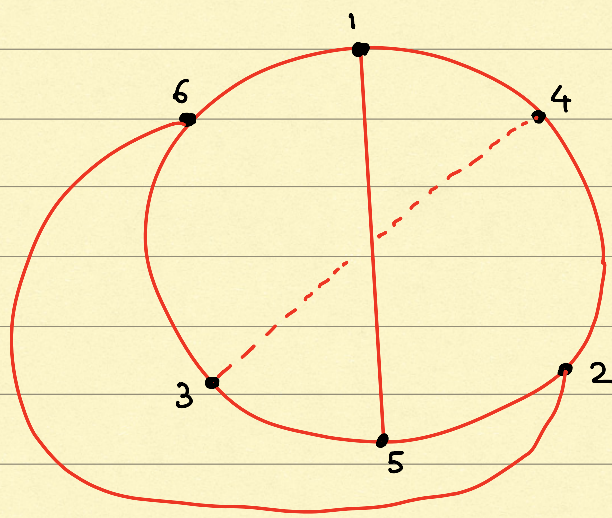
Figure 1



Put (a, d) inside. This forces (b, e) to be outside. Note that (c, g) cannot be drawn inside or outside without crossing either (a, d) or (b, e).

This looks a lot like our previous representation of $K_{3,3}$ namely this.

Figure 2



$K_{3,3}$

The difference essentially is the presence of some extra vertices in Figure 1, namely h and f.

Removing h and f from Figure 1 we get:

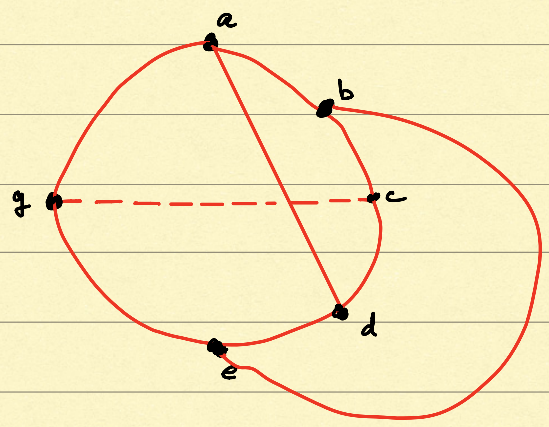
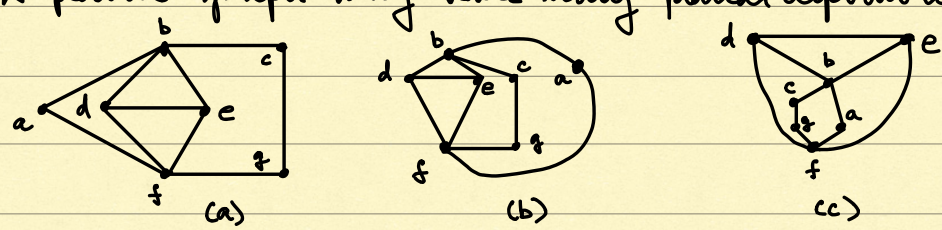


Figure 3

Making the correspondence: $a \longleftrightarrow 1$, $b \longleftrightarrow 6$, $c \longleftrightarrow 3$, $d \longleftrightarrow 5$, $e \longleftrightarrow 2$, $g \longleftrightarrow 4$ we see that Figure 3 is isomorphic to Figure 2. This means Figure 1 (where h & f occur in the middle of edges of Fig 3) is a configuration of $K_{3,3}$.

Euler's Theorem

A planar graph may have many planar depictions.



A graph depicted as a planar graph in three different ways.

The number of regions in the plane determined by the graph remains the same in all three depictions. In the above case it is five. (We count the unbounded region also.)

Let

$\bar{r}$ = # of regions (including the unbounded region)

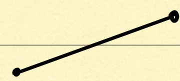
$\bar{e}$ = # of edges

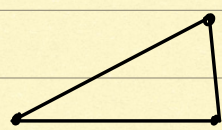
$\bar{v}$ = # of vertices.

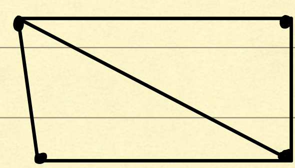
In the above example

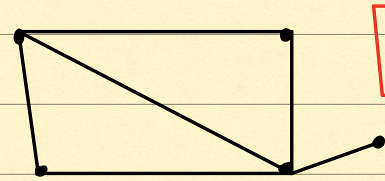
$$\boxed{\bar{r} = 5, \quad \bar{e} = 10, \quad \bar{v} = 7.}$$

<u>More examples</u>

(i)  $\boxed{\bar{r} = 1, \ \bar{e} = 1, \ \bar{v} = 2}$

(ii)  $\boxed{\bar{r} = 2, \ \bar{e} = 3, \ \bar{v} = 3}$

(iii)  $\boxed{\bar{r} = 3, \ \bar{e} = 5, \ \bar{v} = 4}$

(iv)  $\boxed{\bar{r} = 3, \ \bar{e} = 6, \ \bar{v} = 5}$

In every case we see that $\bar{r} - \bar{e} + \bar{v} = 2$.

The following is Euler's Formula.

Theorem: Let $G = (V, E)$ be a connected planar graph. Then for any plane graph depiction of G, we have

$$\boxed{\bar{r} - \bar{e} + \bar{v} = 2}$$

where r is the number of regions in the plane determined by the plane graph depiction, e the number of edges in G, and v the number of vertices in G. In particular the number of regions does not depend on the plane graph depiction of G.

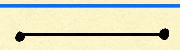
Proof:

By induction on the number of edges.

We will build the plane depiction of G edge by edge.

Let G_n denote the stage with n-edges.

Let r_n, e_n, v_n denote the regions, edges, and vertices of G_n.

$G_1 =$  . Easy to see $\bar{r}_1 - \bar{e}_1 + \bar{v}_1 = 2$.

Suppose we have built G_n and verified that

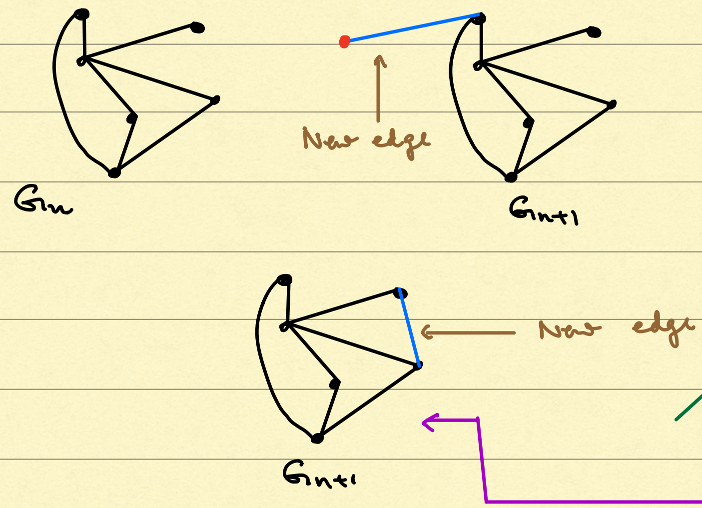
$$\bar{r}_n - \bar{e}_n + \bar{v}_n = 2.$$

We obtain G_{n+1} by adding an edge to G_n so that G_{n+1} is connected.

Either one of the vertices of the new edge is not in G_n or both the vertices of the new edge are in G_n.

The following pictures illustrate what might happen in each case:

Possibilities



In this case $r_{n+1} = r_n$
$e_{n+1} = e_n + 1$
$v_{n+1} = v_n + 1$

Two possibilities for G_{n+1}

In this case
$r_{n+1} = r_n + 1$
$e_{n+1} = e_n + 1$
$v_{n+1} = v_n$.

In case the new edge has a new vertex, then
$r_{n+1} = r_n$, $e_{n+1} = e_n + 1$, $v_{n+1} = v_n + 1$.

In this case
$$r_{n+1} - e_{n+1} + v_{n+1} = r_n - (e_n + 1) + (v_n + 1)$$
$$= r_n - e_n + v_n = 2$$

On the other hand if the new edge has no new vertices then

$$r_{n+1} = r_n + 1, \quad e_{n+1} = e_n + 1, \quad v_{n+1} = v_n,$$

and in this case

$$r_{n+1} - e_{n+1} + v_{n+1} = (r_n + 1) - (e_n + 1) + v_n$$
$$= r_n - e_n + v_n$$
$$= 2.$$

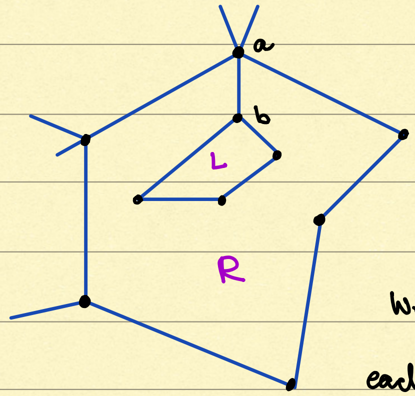
By induction, we are done. q.e.d.

Corollary: If G is a connected planar graph with $e > 1$, then $e \leq 3v - 6$.

Note: This is clearly not true when $e = 0$ or 1. In the corollary we are not allowing the following two graphs:

Proof of Corollary:

Define the degree of a region K determined by a graph to be the number of edges touching K. If K surrounds an edge, the edge is counted twice.



In the picture on the left, the edge (a, b) is counted twice for R.

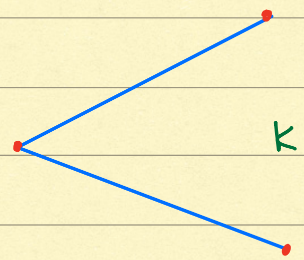
degree of $R = 12$

degree of $L = 4$.

We write $\deg(K)$ for degree of K. Since each edge borders two regions (or is counted twice) we have

$$\sum_{K \text{ region}} \deg K = 2\bar{e}$$

Now each region K has degree ≥ 3 for two edges cannot bound a region. If a region K has only edges bordering it then K surrounds each edge and $\deg K = 4$.



Only 2 edges bordering K means each edge has degree 2 as K surrounds each edge, so deg K = 4. We cannot have only one edge since $\bar{e} > 1$.

So

$$\sum_{K} \deg K \geqslant \sum_{K} 3 = 3\bar{r}$$

Thus

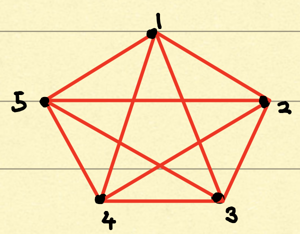
$$2\bar{e} \geqslant 3\bar{r} = 3(\bar{e} - \bar{v} + 2)$$

i.e. $\quad 2\bar{e} \geqslant 3\bar{e} - 3\bar{v} + 6$

$\Rightarrow \quad 3\bar{v} - 6 \geqslant \bar{e}$ as required.

q.e.d.

Showing K_5 is non-planar



K_5

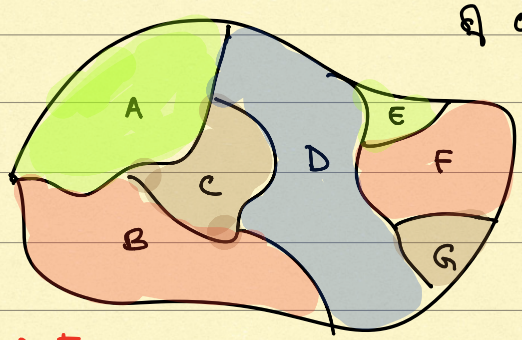
$v = 5, \quad e = 10.$

$3v - 6 = 15 - 6 = 9$, but $e = 10$

So $\quad 3v - 6 \not\geqslant e$.

This means K_5 is not planar.

Map Colouring : Consider the problem of colouring a map of countries drawn on the plane so that no two adjoining countries have the same colour.



The four colour conjecture

The question is : Can we always do it in 4 colours?

Through much of the 20th century mathematicians tried to prove that four colours are enough. In 1976 Appel and Haken showed that 4 colours do indeed suffice. There were 1955 cases they considered, each involved numerous subcases. Computers generated these cases .... We will not prove this !!

You can think of the map as a graph with borders as edges and vertices where the edges meet.

There is another depiction via the dual graph. This is more useful. In the dual graph one has a vertex for each country and drawing an edge between two vertices if the corresponding countries share a common border. The dual graph of the map shown is