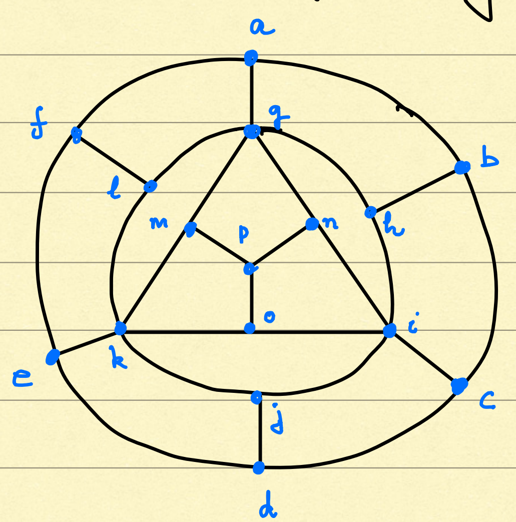
Jan 18, 2018

Lecture 5

Example : Is the following graph bipartite?



Use the procedure given in the proof of the theorem. Start with a, and put it on the left. Put vertices joined by even-length paths to a to the left and those joined by odd-length paths to a to the right. If the new representation of the graph has any edge joining two vertices on the left (or on the right) then the graph is not bipartite. Otherwise it is.

In this case, it is bipartite. See Figure 1-16 on p.28 of the text book.

Section 1-4 on the next page.

§1.4 Planar graphs:

Definition: A graph is *planar* if it can be drawn on a plane without edges crossing.

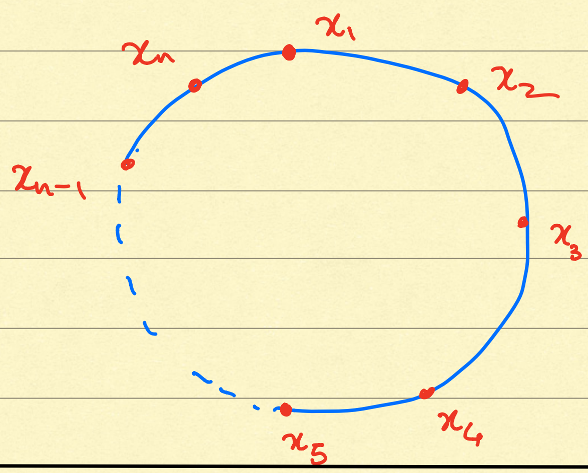
A planar description of a graph is called a *plane graph*.

THE CIRCLE-CHORD METHOD:

How does one know if a graph is planar?

cross out.

Suppose a graph $G=(V,E)$ is a connected ~~planar~~ graph and we can find a circuit $C = x_1 - x_2 - \dots - x_n - x_1$ containing every element of V. Draw it as circle with the vertices x_i marked off as in the picture.



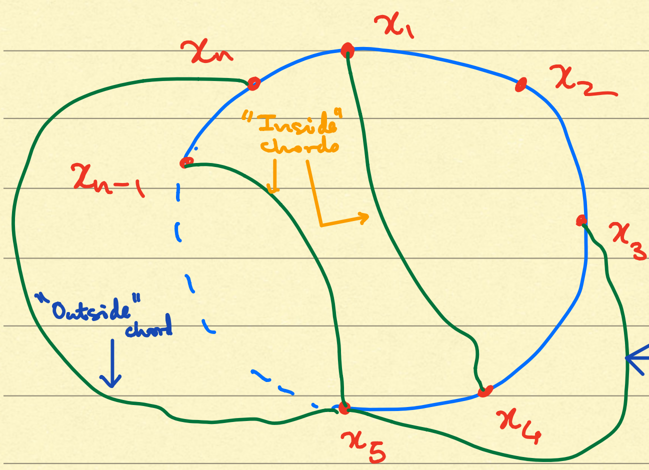
Assumption

> Every vertex of G is a vertex of C, i.e.
> $$V = \{x_1, x_2, \dots, x_n\}.$$

(Given an arbitrary graph, it is by no means certain that such a circuit can be found, but in the examples and homework problems deciding the planarity of a graph, such a circuit will be there. You may have to look to find a C.)

Coming back to our circuit C. By assumption every edge is incident upon to vertices in C.

A chord is an edge not in C drawn in such a way that it meets our circle only at the end-points.



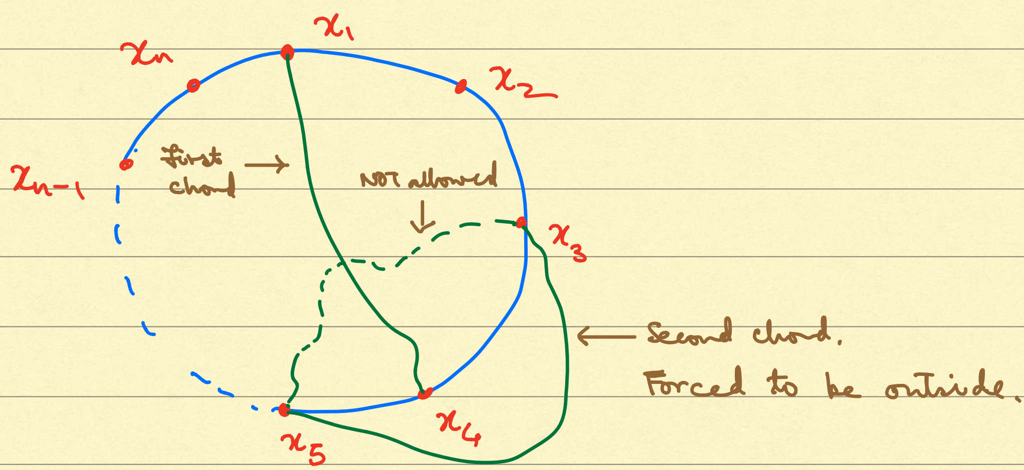
The blue circuit is C.

Start with the C. Draw a chord. One has a choice; one can either draw the chord inside the circle or outside the circle. Once a choice has been made, say for definiteness that the chord is drawn inside, then this forces certain other choices. For example in the picture above, if (x_1, x_4) is chosen to be an "inside" chord, (x_3, x_5) is forced to be an "outside" chord, and if (x_1, x_4) was drawn outside C then (x_3, x_5) would be forced to be inside C.

So keep drawing chords so that they don't cross each other. If at some point a new chord is forced to cross an earlier chord whether the new chord is an inside chord or an outside chord, then we claim the graph is non-planar! If not, it is clearly planar since we have drawn a rep^n of G so that no two edges cross.

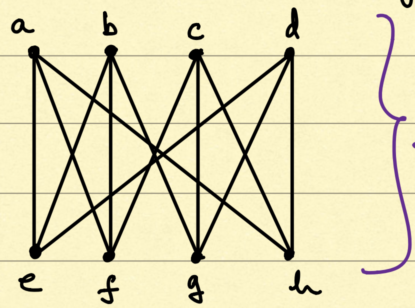
To prove the claim, first note that the notion of inside and outside is relative. We may imagine the circle to the equator of the earth. A chord then has end-points on the equator and the rest of it is either entirely in the southern hemisphere or entirely in the northern hemisphere. We can draw the chords so that they do not pass through the poles. Whether the chord is inside the circle (= equator) or outside depends entirely upon one's point of view.

So it does not matter how we draw our first chord — inside or outside. This more or less forces us to draw other chords in a certain way (inside or outside) if we do not wish to cross previous chords. If the graph is planar none of the chords cross, and vice-versa.



This shows that if, during the process of drawing chords systematically, if at some point a new chord crosses some older chords whether we draw the new chord on the inside or outside, then the graph is <u>non-planar</u>. This proves the claim. Conversely if all chords can be drawn without crossing each other, G is <u>by definition</u> a planar graph.

Example: Consider the graph

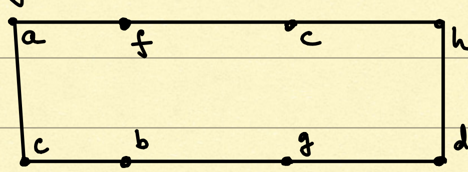


Planar or non-planar? Use Circle-chord method.

<u>Step 1</u>: See if we can find circuit which contains all the vertices. There is one, namely

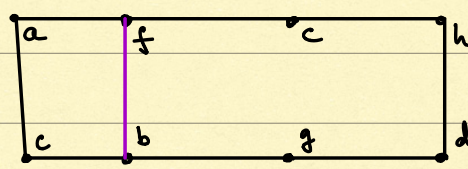
$$C = a - f - c - h - d - g - b - e.$$

Now re-arrange! Here is C drawn as a "circle".



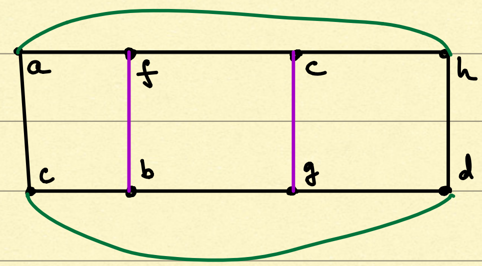
May not be the only circuit containing all vertices!

Draw the first edge (f,b) "inside".



This forces (a,h) and (c,d) to be "outside" and (c,g) to be "inside".

The resulting picture is:



It is clearly planar.

Next time we will use the same circle-chord method to show that a certain bipartite graph called $K_{3,3}$ is non-planar. $K_{3,3}$ is the bipartite graph $G=(V,E)$ with six vertices such that V_1 and V_2 each have three vertices, and every vertex in V_1 is adjacent to every vertex in V_2 and vice-versa.