Jan 16, 2018

Lecture 4

Example 1: Suppose we want to construct a graph with 20 edges and have every vertex of degree 4. How many vertices must the graph have?

Answer: Let $G=(V,E)$ be such a graph.

$$\sum_{v\in V} \deg(v) = 2\,(\#\text{ of edges})$$

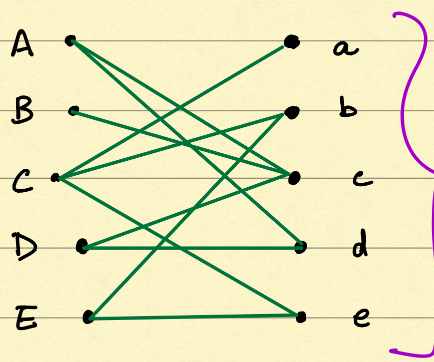
i.e.

$$\sum_{v\in V} (4) = 2\,(20)$$

i.e. $\quad 4\,(\#\text{ of vertices}) = 40$

$\Rightarrow \quad \#\text{ of vertices} = 10 \quad \longleftarrow$ ANSWER.

Bipartite graphs: Recall the graph we had for matching jobs to qualified people (see Lecture 1)

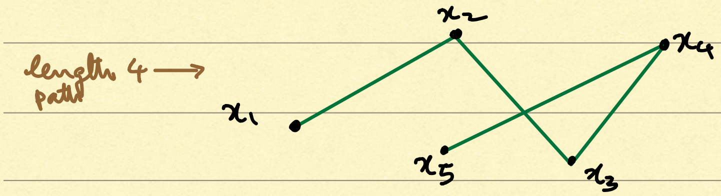


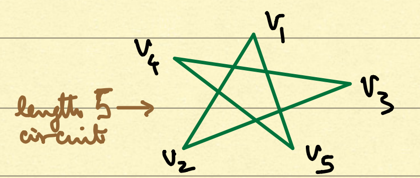
This is an example of a bipartite graph (see definition below).

Definition: A graph $G=(V,E)$ is bipartite if $V = V_1 \cup V_2$ with V_1 and V_2 non-empty, disjoint (i.e., $V_1 \cap V_2 = \phi$) and every edge joins a vertex in V_1 with a vertex in V_2.

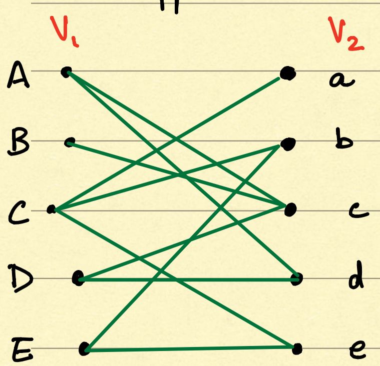
Recall that a <u>circuit</u> in a graph G is a sequence of vertices written $x_1 - x_2 - x_3 - \cdots - x_n - x_1$ such that $x_1, \ldots, x_n$ are distinct and two consecutive edges are adjacent (including x_1 and x_n).

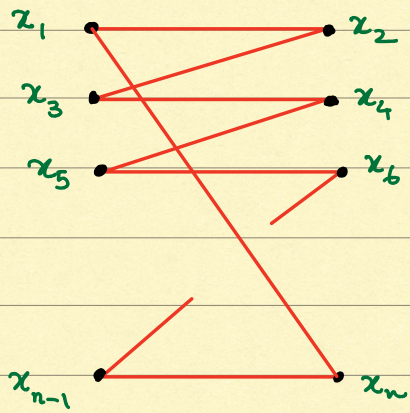
The <u>length</u> of a path or circuit is the number of edges linking successive vertices. Thus a path of the form $P = x_1 - x_2 - \cdots - x_n$ has length $n-1$, and circuit $C = v_1 - v_2 - v_3 - \cdots - v_n - v_1$ has length n.





Suppose $G = (V, E)$ is bipartite, say $V = V_1 \cup V_2$, with V_1, V_2 disjoint and non-empty. Put all the vertices in V_1 in a column on the left and all the vertices of V_2 in a column on the right. If $x_1 - x_2 - \cdots - x_n - x_1$ is a circuit in G then assuming x_1 is on the left, then x_2 is on the right, x_3 on the left, and we have alternately a left vertex and a right vertex (if x_1 occurs on the right, once again we have alternately a right vertex followed by a left vertex, followed by a right vertex).



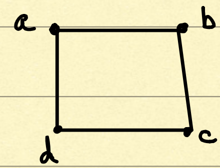


Since we assumed x_1 is on the left, it is clear that all the <u>odd</u> subscript vertices x_i are on the left and all the <u>even</u> subscript vertices x_i are on the right.

Since x_n is adjacent to x_1, it must be on the right. This means n is even.

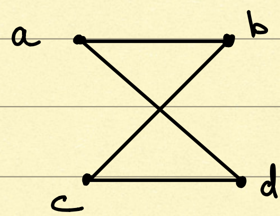
<u>Question</u>:

Is



bipartite?

Re-draw as



Clearly bipartite!
$V_1 = \{a, c\}$, $V_2 = \{b, d\}$.

<u>Defn</u>: A <u>component</u> of a graph G is a connected subgraph of G such that there is no path between any vertex in H and any vertex of G not in H.

Every vertex belongs to a unique component. If x is a vertex, consider all other vertices which can be reached from x by a path in G. All such vertices and x, together with the edges of G connecting them, forms a component, the only component containing x. The graph G partitions into components.

<u>Theorem</u> (Thm 2 of §1.3 of book). A graph $G=(V,E)$ is bipartite if and only if every circuit in G has even length.

<u>Proof</u>: We have already seen the "only if" part, for we showed that a circuit in a bipartite graph must have even length.

Now for the "if" part, i.e., we have to show that if every circuit in G has even length then G is bipartite. It is enough to prove this for connected G. If we show each component is bipartite, then it is clear that G is bipartite.

Thus, without loss of generality, we may assume that G is connected. Take any vertex, call it a, and put it on the left. Put all vertices adjacent to a on the right.

Next put all vertices which are two edges away from a on the left. Such vertices are exactly those at the end of paths of length 2 starting from a. In general, if there is a path of odd length between a and a vertex x, then put x on the right, and if there is a path of even length connecting

a to x, put x on the left. Since G is connected there is always at least one path connecting a and x.

Issue: What if there are two or more different paths connecting a and x? Can one be of even length and another of odd length.

Say P and Q are paths starting at a and ending at x. Let Q^{-1} be the return path for Q, starting from x, ending at a. Then P followed by Q^{-1} is a circuit starting and end at a. This circuit is denoted PQ^{-1} ("P followed by Q^{-1}").

Now,

$$\text{length}(PQ^{-1}) = \text{length}(P) + \text{length}(Q).$$

The left side of the above equation is even, since our hypothesis is that every circuit in G is of even length. Therefore the right side is even. It follows that either both the terms on the right side are odd or both sides are even.

This means the issue raised creates no problem and we have a well defined way of putting x on the left or on the right.

Can one have an edge between two vertices on the left?

We will argue this is not possible.

Suppose (x,y) is an edge with both x and y on the left. Choose a path

$$P = v_1 - \dots - v_n$$

joining a to x (so $v_1 = a$, $v_n = x$), and a path

$$Q = w_1 - \dots - w_m$$

joining a to y (so $w_1 = a$, $w_m = y$). Note that

$$\text{length}(P) = n-1$$

$$\text{length}(Q) = m-1,$$

and since length P, length Q are even, n and m are both <u>odd</u>. Let R be the circuit obtained by first using path P, then the edge (x,y), and finally Q^{-1}. Then,

$$R = \underbrace{\overset{a}{\overset{\shortparallel}{v_1}} - \dots - \overset{x}{\overset{\shortparallel}{v_n}}}_{P} - \underbrace{\overset{y}{\overset{\shortparallel}{w_m}} - w_{m-1} - \dots - \overset{a}{\overset{\shortparallel}{w_1}}}_{Q^{-1}}$$

is a circuit of length $n+m-1$. Since n and m are odd, $n+m$ is even, and so $n+m-1$ is odd. Since R is a circuit, by our hypothesis this is not possible. Thus there is no edge between the two vertices on the right.

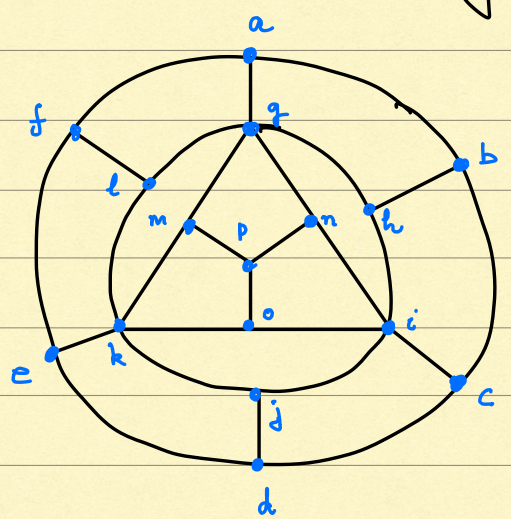
Similarly there is no edge between two vertices on the right.

In other words, G is bipartite.

q.e.d.

Here is a sneak preview of some bits of the next lecture!

Example : Is the following graph bipartite?



Use the procedure given in the proof of the theorem.
Start with a, and put it on the left. Put vertices joined by even-length paths to a to the left and those joined by odd-length paths to a to the right. If the new representation of the graph has any edge joining two vertices on the left (or on the right) then the graph is not bipartite. Otherwise it is.

In this case, it is bipartite. See Figure 1-16 on p.28 of the text book.

Section 1-4 on the next page.

§1.4 Planar graphs:

Definition: A graph is planar if it can be drawn on a plane without edges crossing.

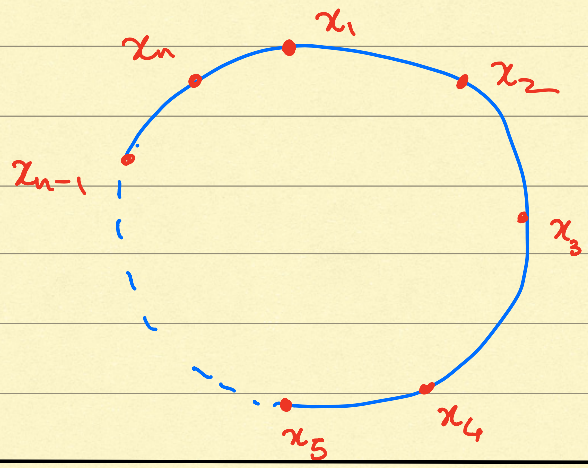
A planar description of a graph is called a plane graph.

THE CIRCLE-CHORD METHOD:

How does one know if a graph is planar?

Suppose a graph $G=(V,E)$ is a connected ~~planar~~ graph and we can find a circuit $C = x_1 - x_2 - \cdots - x_n - x_1$ containing every element of V. Draw it as circle with the vertices x_i marked off as in the picture.

cross out.



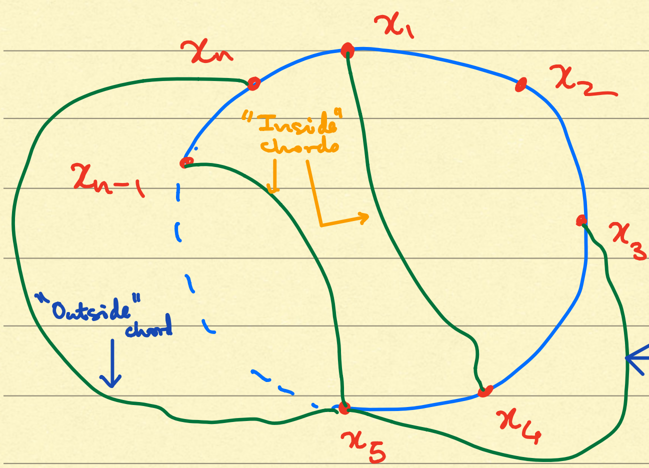
Assumption

$$\boxed{\text{Every vertex of } G \text{ is a vertex of } C, \text{ i.e. } V = \{x_1, x_2, \ldots, x_n\}.}$$

(Given an arbitrary graph, it is by no means certain that such a circuit can be found, but in the examples and homework problems deciding the planarity of a graph, such a circuit will be there. You may have to look to find a C.)

Coming back to our circuit C. By assumption every edge is incident upon to vertices in C.

A chord is an edge not in C drawn in such a way that it meets our circle only at the end-points.



The blue circuit is C.

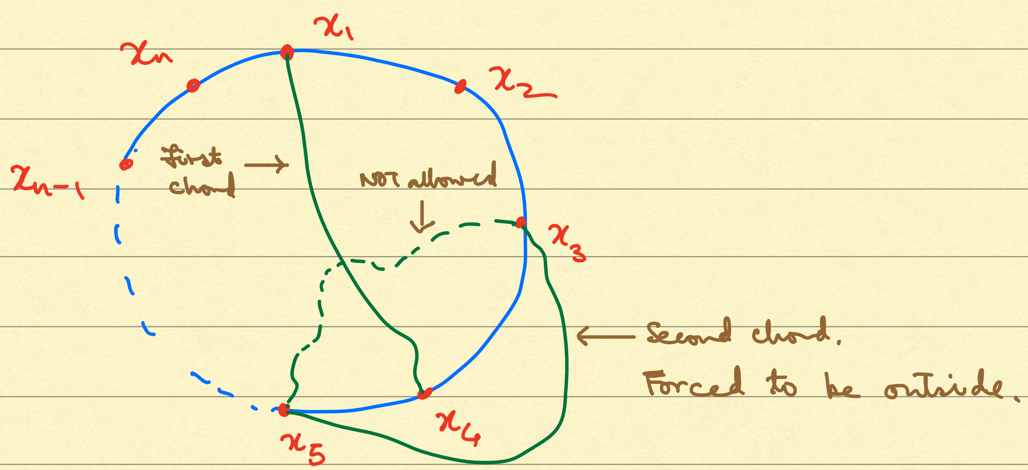
"Outside" Chord

Start with the C. Draw a chord. One has a choice; one can either draw the chord inside the circle or outside the circle. Once a choice has been made, say for definiteness that the chord is drawn inside, then this forces certain other choices. For example in the picture above, if (x_1, x_4) is chosen to be an "inside" chord, (x_3, x_5) is forced to be an "outside" chord, and if (x_1, x_4) was drawn outside C then (x_3, x_5) would be forced to be inside C.

So keep drawing chords so that they don't cross each other. If at some point a new chord is forced to cross an earlier chord whether the new chord is an inside chord or an outside chord, then we claim the graph is non-planar! If not, it is clearly planar since we have drawn a rep^{n} of G so that no two edges cross.

To prove the claim, first note that the notion of inside and outside is relative. We may imagine the circle to the equator of the earth. A chord then has end-points on the equator and the rest of it is either entirely in the southern hemisphere or entirely in the northern hemisphere. We can draw the chords so that they do not pass through the poles. Whether the chord is inside the circle (= equator) or outside depends entirely upon one's point of view.

So it does not matter how we draw our first chord — inside or outside. This more or less forces us to draw other chords in a certain way (inside or outside) if we do not wish to cross previous chords. If the graph is planar none of the chords cross, and vice-versa.



This shows that if, during the process of drawing chords systematically, if at some point a new chord crosses some older chords whether we draw the new chord on the inside or outside, then the graph is <u>non-planar</u>. This proves the claim. Conversely if all chords can be drawn without crossing each other, G is by definition a planar graph.