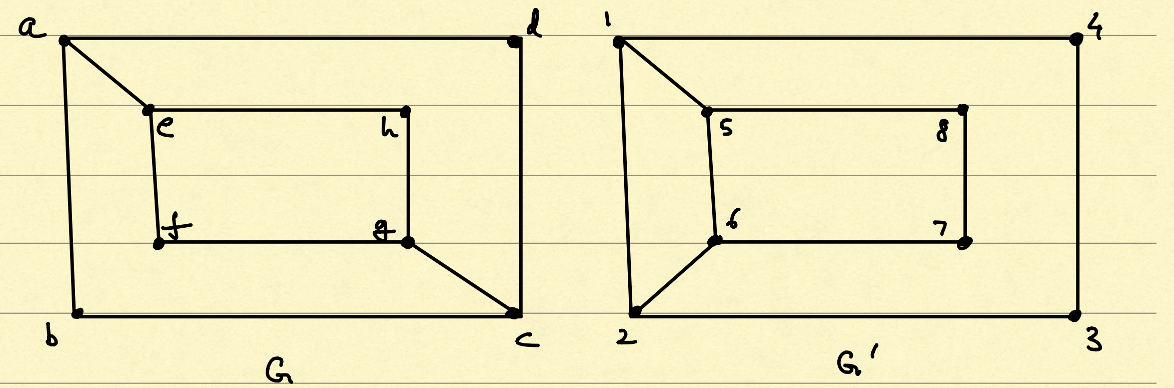
Jan 11, 2018 Lecture 3

**Example** : Are the two graphs




isomorphic ?

# of vertices in G = # vertices in G' (=8)

# of edges in G = # of edges in G'. (=10)

# of degree 2 vertices in G = # of degree 2 vertices in G' (=4)

# of degree 3 vertices in G = # of degree 3 vertices in G' (=4)

Let $S = (V_1, E_1)$ be the subgraph of $G \overset{(V,E)}{=}$ consisting of degree 2 vertices of V and all the edges between them.

Let $S' = (V_1', E_1')$ be the subgraph of $G' = (V', E')$ described in the same way, i.e, consisting of all vertices of G' of degree 2 and all the edges between them.

$$V_1 = \{b, d, f, h\}, \quad E_1 = \phi \longleftarrow \text{the empty set}$$

$$V_1' = \{3, 4, 8, 7\}, \quad E_1' = \{(7,8), (3,4)\}$$

If G and G' are isomorphic, then S and S' are also isomorphic. But clearly S and S' are NOT isomorphic.

In fact $E_1$ is empty whereas $E_1'$ is not.
This means $G$ and $G'$ are not isomorphic.

Defn: A graph $G = (V, E)$ is said to be complete if every pair of distinct vertices in $G$ are adjacent.

Note: If $G$ is a complete graph and has $n$ vertices then $G$ is isomorphic to $K_n$.

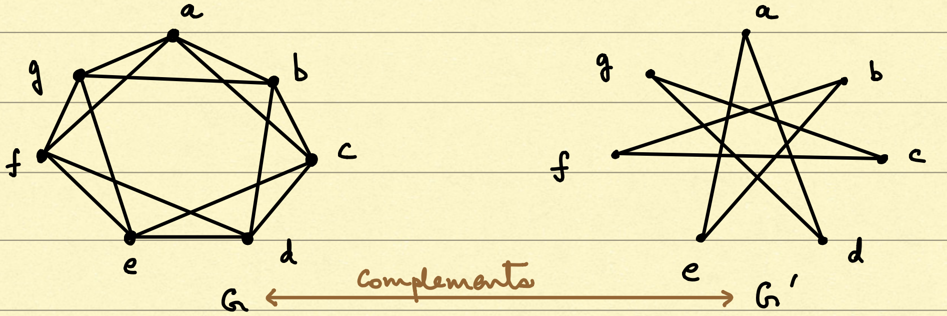
The complement of a graph:

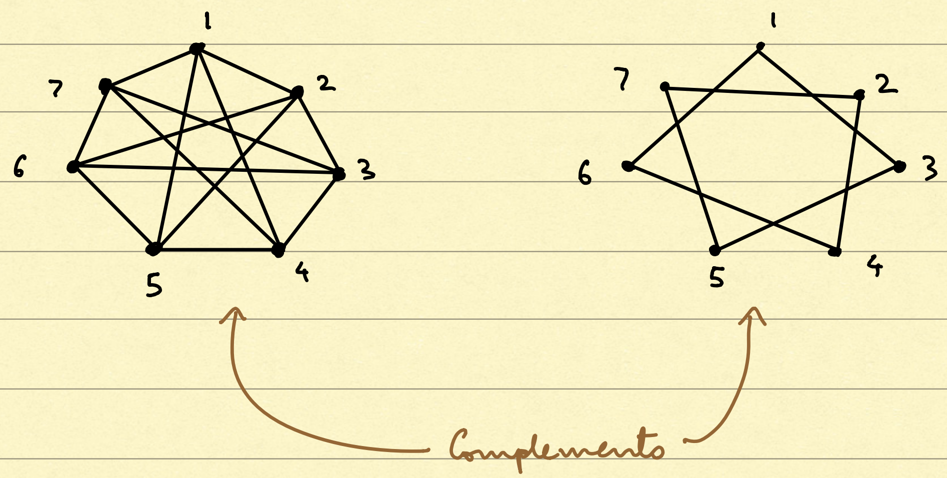
Let $G = (V, E)$ be a graph. Its complement $\bar{G}$ is a graph whose set of vertices is the same as that of $G$, and whose set of edges $\bar{E}$ are exactly those pair of distinct vertices which are not adjacent in $G$.

$$\bar{G} = (V, \bar{E}).$$

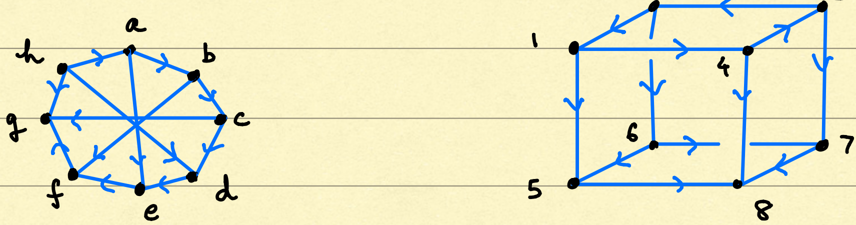
The union of $E$ and $\bar{E}$ consists all pairs of distinct vertices in $V$.

The graph $G^* = (V, E \cup \bar{E})$ is clearly a complete graph.






**Read:** Example 3, page 18, on isomorphisms of directed graphs.




Are the above two directed graphs isomorphic? For answering this you need the notion of in-degree and out-degree of a vertex in a directed graph. The in-degree of $v \in V$ is the number of edges ending at $v$, whereas the out-degree of $v$ is the number of edges starting at $v$.

Section 1.3 on next page.

## §1.3 Edge Counting:

Formula for counting edges.

**Theorem 1**: In any graph, the sum of the degrees of all vertices is equal to twice the number of edges.

Proof:

Let $G = (V, E)$ be a graph.

For $v \in V$, $\deg v$ counts all the edges incident on $v$, and so the sum of the degrees of all vertices counts the number of instances of an edge being incident on a vertex.

Now, each edge is incident on two vertices.

This means that the number of instances of an edge being incident on a vertex is twice the number of edges.

q.e.d.

**Remark**: The sum of degrees is therefore an even number! The following is an immediate consequence.

**Corollary**: If $G = (V, E)$ is a graph, then the number vertices of odd degree is even.

Proof: $\displaystyle\sum_{\deg v \text{ odd}} \deg(v) = \sum_{v \in V} \deg(v) - \sum_{\deg v \text{ even}} \deg(v).$

The right side is the difference of two even numbers, and so is even. This means the left side is even. q.e.d.

<u>Example 1</u> : Suppose we want to construct a graph with 20 edges and have every vertex of degree 4. How many vertices must the graph have?

<u>Answer</u> : Let $G=(V,E)$ be such a graph.

$$\sum_{v \in V} \deg(v) = 2(\#\text{ of edges})$$

i.e.

$$\sum_{v \in V} (4) = 2(20)$$

i.e. $4(\#\text{ of vertices}) = 40$

$\Rightarrow \#\text{ of vertices} = 10 \longleftarrow$ ANSWER.

<u>Example 2</u> : Is it possible to have a group of eleven people such that each person knows exactly five other persons in the group?

<u>Soln</u> : Can model the situation by a graph, with vertices for each person in the group and edges representing the fact that the two persons at the ends of an edge know each other. Then each vertex has degree 5, and there are eleven vertices. This means there are 11 vertices of odd degree, which violates the Corollary. So such a graph is NOT POSSIBLE.