March 29, 2018   Lecture 22

In the last lecture we gave the formula

$$\sqrt{1-4x} = 1 - 2\sum_{n=0}^{\infty} \frac{(2n)!}{n!\,(n+1)!}\, x^{n+1} \quad \text{———} (*)$$

and indicated how one might prove it. Here are the details.

We have $\frac{d}{dx} y = \frac{1}{2}(1+y)^{-1/2}$.

The higher derivatives are

$$\left.\begin{aligned} \frac{d^n}{dy^n}(1+y)^{1/2} &= \tfrac{1}{2}\left(\tfrac{1}{2}-1\right)\left(\tfrac{1}{2}-2\right)\cdots\left(\tfrac{1}{2}-n+1\right)(1+y)^{\frac{1}{2}-n} \\ &= \tfrac{1}{2}\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\cdots\left(\frac{-2n+3}{2}\right)(1+y)^{\frac{1}{2}-n} \\ &= \frac{(-1)^{n-1}}{2^n}(1)(3)\cdots(2n-3)(1+y)^{1/2-n} \end{aligned}\right\} \quad n \geq 2$$

Multiply and divide by $(2)(4)(6)\ldots(2n-2)$

$$\frac{d^n}{dy^n}(1+y)^{1/2} = \frac{(-1)^{n-1}}{2^n}(1)(3)(5)\cdots(2n-3)\cdot\frac{(2)(4)(6)\cdots(2n-2)}{(2)(4)(6)\cdots(2n-2)}(1+y)^{1/2-n}$$

$$\frac{d^n}{dy^n}(1+y)^{1/2} = \frac{(-1)^{n-1}}{2^n}\,\frac{(2n-2)!}{2^{n-1}(n-1)!}\,(1+y)^{1/2-n}$$

$$= 2\,\frac{(-1)^{n-1}}{4^n}\,\frac{(2n-2)!}{(n-1)!}\,(1+y)^{1/2-n}, \quad n \geq 1$$

The formula also works for $n=1$ as can be checked. ($n=0$ also OK)

This means that the Taylor's series expansion of $(1+y)^{1/2}$ is

$$(1+y)^{1/2} = 1 + 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4^n} \frac{(2n-2)!}{(n-1)!} \frac{y^n}{n!}$$

Making the change of variables $m+1 = n$ we get

$$(1+y)^{1/2} = 1 + 2\sum_{m=0}^{\infty} \frac{(-1)^m}{4^{m+1}} \frac{(2m)!}{m!\,(m+1)!} y^{m+1}$$

Now set $y = -4x$ to get

$$(1-4x)^{1/2} = 1 + 2\sum_{m=0}^{\infty} \frac{(-1)^m}{4^{m+1}} \frac{(2m)!}{m!\,(m+1)!} (-1)^{m+1} 4^{m+1} x^{m+1}$$

$$= 1 - 2\sum_{m=0}^{\infty} \frac{(2m)!}{m!\,(m+1)!} x^{m+1}$$

This proves $(*)$.

We also had the functional relation for the (last recurrence eqn of the last lecture)

$$x g(x)^2 - g(x) + 1 = 0$$

together with the initial condition $g(0) = 1$. From the functional eqn one gets $g(x) = \left(1 \pm \sqrt{1-4x}\right)/2x$. Let us show (using $g(0)=1$) that $g(x) = \left(1 - \sqrt{1-4x}\right)/2x$ (and not the other answer).

Now $$\lim_{x\to 0} \frac{\left(1-\sqrt{1-4x}\right)}{2x} = \lim_{x\to 0} \frac{\frac{d}{dx}\left(1-\sqrt{1-4x}\right)}{\frac{d}{dx}(2x)} \quad \text{(L'Hopital)}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{2}(1-4x)^{-1/2}(-4)}{2}$$

$$= \lim_{x\to 0} (1-4x)^{-1/2}$$

$$= 1.$$

Note that $\lim_{x\to 0} \frac{(1+\sqrt{1-4x})}{2x} = \infty$. Since $g(0)$ is 1 and not ∞, we must have

$$g(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

(<u>Note</u>: $g(x)$ is continuous at $x=0$ because it has a power series expansion around $x=0$).

<u>Example</u> (Placing parenthesis) Let a_n be the number of ways to place parentheses to multiply the n numbers $k_1 \times k_2 \times \dots \times k_n$. Find a recurrence relation for $\{a_n\}$.

This is what is meant by the "number of ways to place parentheses. When $n=2$ we have $k_1 \times k_2$ and only one way to parenthesize in order to multiply, namely $(k_1 \times k_2)$. When $n=3$, we have to see how to place parentheses in $k_1 \times k_2 \times k_3$. The only ways are $((k_1 \times k_2) \times k_3)$ and $(k_1 \times (k_2 \times k_3))$. Thus $a_2 = 1$ and $a_3 = 2$.

For simplicity, to make the recurrence work, set $a_1 = 1$, even though it is unclear what meaning to assign to a_0 and a_1.

Let $n \geq 2$. Look at the last multiplication — the outer most parentheses. There is an i between 1 and n, such that the last multiplication is of the form

$$(k_1 \times \dots \times k_i) \times (k_{i+1} \times \dots \times k_n) \qquad \text{------} (*)$$

with some way of multiplying together (two at a time) each of the products in parentheses. There are a_i ways of placing parentheses for $k_1 \times k_2 \times \dots \times k_i$ and a_{n-i} ways of placing parentheses for $k_{i+1} \times \dots \times k_n$. Thus if the last multiplication is of the form (*) then there are $a_i a_{n-i}$ ways of parenthesizing. Note that i lies between 1 and $n-1$. Summing we get

$$a_n = \sum_{i=1}^{n-1} a_i a_{n-i} \quad , \quad n \geq 2.$$

This is very close to the recurrence relation we had in the last example of the last lecture and one can solve it by similar means. The details are left to you.