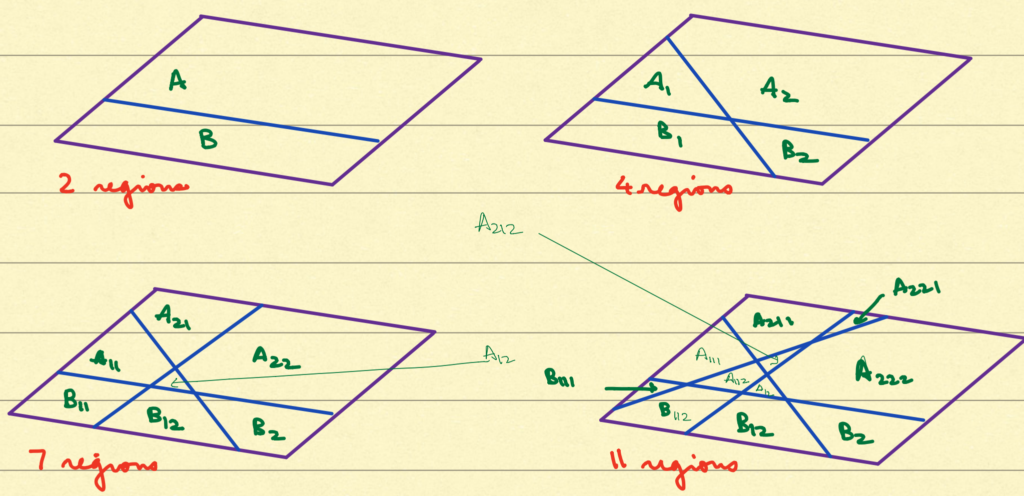
March 27, 2018 Lecture 21

Recall that in the Tower of Hanoi game, the number of moves a_n required to shift n pegs from peg A to peg C satisfies the relation

$$a_n = 2a_{n-1} + 1 \qquad n \geq 2.$$

Such a relation, relating a_n to preceding a_i is called a recurrence relation. Here are two examples:

1. Dividing the plane: Suppose we draw n distinct lines on a plane so that every pair of lines intersect (i.e, they are not parallel) and no three lines intersect in a point. These lines divide the plane into various regions, say a_n in number.



It is easy to see that $a_1 = 2$, $a_2 = 4$, $a_3 = 7$, $a_4 = 11$. In general the argument is as follows. Suppose

$n-1$ lines have been drawn according to the rules given. Now draw the n^{th} line. Call it l_n. Imagine a moving particle on the line — moving in the increasing x direction, unless l_n is vertical, in which case move in the increasing y direction. We start the particle from a point on l_n such that there are no points to the left of this initial point. Label the first line (amongst the earlier $n-1$ lines) it encounters l_1, the second as $l_2, \ldots,$ the last as l_{n-1}. Clearly l_n passes through n of the a_{n-1} regions $l_1, \ldots, l_{n-1}$ divide the plane into: the region before encountering l_1, the region when the particle is between l_1 and l_2, $\ldots$, the region after crossing l_{n-1}. Each of these n regions get divided into two creating n new regions. This means

$$a_n = a_{n-1} + n, \qquad n \geq 0.$$

In this case the solution is easy.

$$a_1 - a_0 = 1$$

$$a_2 - a_1 = 2$$

$$\vdots$$

$$a_{n-1} - a_{n-2} = n-1$$

$$a_n - a_{n-1} = n$$

Add. Get

$$a_n - a_0 = 1+2+\ldots+n = \binom{1}{1} + \binom{2}{1} + \ldots + \binom{n}{1} = \binom{n+1}{2} = \frac{n(n+1)}{2}$$

Since $a_0 = 1$, $\quad a_n = 1 + \frac{n(n+1)}{2}$.

Read other examples from §7.1 by yourself. They are all fairly easy.

Section 7.3 Solutions of Linear Recurrence Relations

Consider recurrence relations of the following kind.

$$\boxed{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_r a_{n-r}} \qquad (*)$$

where r is a fixed integer and $c_1, c_2, \dots, c_r$ are constants. Relations like $(*)$, which hold for all sufficiently large n, are called <u>linear homogeneous recurrence relations</u>

The solutions for $(*)$ are often of the form $a_n = \alpha^n$ (in fact they are linear combinations of such and related expressions). If one substitutes α^k for a_k in $(*)$ one gets (for $n \geq r$)

$$\alpha^n = c_1 \alpha^{n-1} + c_2 \alpha^{n-2} + \dots + c_r \alpha^{n-r}, \qquad n \geq r.$$

i.e.

$$\boxed{\alpha^r - c_1 \alpha^{r-1} - c_2 \alpha^{r-2} \dots - c_{r-1}\alpha - c_r = 0}$$

This is the so called <u>characteristic equation</u> for $(*)$.

<u>The space of solutions for $(*)$</u>: Consider the linear homogeneous recurrence relation $(*)$. If $\{f_n\}$ and $\{g_n\}$ are two solutions of $(*)$ (i.e., $\forall\ n \geq r$, $f_n = c_1 f_{n-1} + \dots + c_{r-1} f_{n-r+1} + c_r f_{n-r}$ and $g_n = c_1 g_{n-1} + \dots + c_{r-1} g_{n-r-1} + c_r g_{n-r}$) then so is

any linear combination $c\{f_n\}+d\{g_n\}$ of $\{f_n\}$ and $\{g_n\}$ as is easily verified by substitution. Here c and d are constants. This means the space of solutions is a vector space S. Note that $f_n, n\geq 0$ are completely determined by $f_0, f_1, \ldots, f_{r-1}$. Indeed if we know $f_0, f_1, \ldots, f_{r-1}$ then we can find f_r by the recurrence relation $(*)$, and using $f_1, f_2, \ldots, f_r$, we can find f_{r+1} and so on. This means there is a one-to-one correspondence between S and $\mathbb{R}^r$, the map being $\{f_n\} \longmapsto \begin{pmatrix} f_0 \\ \vdots \\ f_{r-1}\end{pmatrix}$. This is a linear map. Hence S and $\mathbb{R}^r$ are isomorphic. It follows that the space of solutions S for $(*)$ is an r-dimensional vector space.

Distinct roots Suppose the characteristic equation for $(*)$

$$\alpha^r - c_1\alpha^{r-1} - c_2\alpha^{r-2} - \cdots - c_r = 0$$

has r distinct roots $\alpha_1, \ldots, \alpha_r$. Now $\{\alpha_1^n\}, \{\alpha_2^n\}, \ldots, \{\alpha_r^n\}$ are solutions of $(*)$ as are all their linear combinations. Can there be other solutions? If the r solutions $\{\alpha_1^n\}, \ldots, \{\alpha_r^n\}$ are linearly independent then the space V spanned by them is r dimensional. Since V is a subspace of S and S is r dim'l, it follows $V=S$, i.e., every solution of $(*)$ is a linear combination of $\alpha_1^n, \alpha_2^n, \ldots, \alpha_r^n$. Using the Van der Monde determinant it is easy to see that $\{\alpha_1^n\}, \ldots, \{\alpha_r^n\}$ a linearly independent. Indeed suppose we have constants $d_1, \dots, d_r$ such that

$$d_1 \alpha_1^n + d_2 \alpha_2^n + \dots + d_r \alpha_r^n = 0 \qquad \forall\, n \geq 0.$$

Then

$$\begin{bmatrix} \alpha_1^0 & \alpha_2^0 & \dots & \alpha_r^0 \\ \alpha_1^1 & \alpha_2^1 & \dots & \alpha_r^1 \\ \alpha_1^2 & \alpha_2^2 & \dots & \alpha_r^2 \\ \vdots & \vdots & & \vdots \\ \alpha_1^{r-1} & \alpha_2^{r-1} & \dots & \alpha_r^{r-1} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

(Here 0^0 is regarded as 1, so if any $\alpha_j = 0$, then $\alpha_j^0 = 1$)

Since the determinant of the coeff matrix is the Van der Monde determinant, i.e., $\prod_{i<j} (\alpha_i - \alpha_j)$, we see that the determinant is nonzero and hence $d_1 = d_2 = \dots = d_r = 0$.

Let $f(x) = x^r - c_1 x^{r-1} - c_2 x^{r-2} - \dots - c_{r-1} x - c_r$.

Suppose α is a multiple root of $f(x)$. Then it is a multiple root of $x^{n-r} \cdot f(x) = 0$ for every $n \geq r$. Fix $n \geq r$. Let $g(x) = x^{n-r} f(x)$. Then

$$g(x) = x^n - c_1 x^{n-1} - c_2 x^{n-2} - \dots - c_{r-1} x^{n-r+1} + c_r x^{n-r},$$

and since α is a multiple root of $g(x) = 0$, we must have $g'(\alpha) = 0$. This means

$$n\alpha^{n-1} = (n-1) c_1 \alpha^{n-2} + (n-2) c_2 \alpha^{n-3} + \dots + (n-r+1) c_{r-1} \alpha^{n-r} + (n-r) c_r \alpha^{n-r-1}$$

Multiply the above by α. Get

$$n\alpha^n = c_1 \left[(n-1)\alpha^{n-1}\right] + c_2 \left[(n-2)\alpha^{n-2}\right] + \dots + c_{r-1}\left[(n-r+1)\alpha^{n-r+1}\right] + c_r \left[(n-r)\alpha^{n-r}\right]$$

It follows that $\{n\alpha^n\}$ is a solution of $(*)$.

The same sort of reasoning shows that if α is a root of $f(x)$ of multiplicity m then $\{\alpha^n\}, \{n\alpha^n\}, \{n^2\alpha^n\}, \ldots, \{n^{m-1}\alpha^n\}$ are all solutions of $(*)$. Moreover, one can show they are linearly independent.

In fact if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are distinct non-zero roots of $f(x)=0$ with α_i of multiplicity m_i (this forces $m_1+m_2+\ldots+m_k=r$, since $f(x)$ has degree r) then the r solutions

$$\{\alpha_1^n\}, \{n\alpha_1^n\}, \ldots, \{n^{m_1-1}\alpha^n\}; \{\alpha_2^n\}, \{n\alpha_2^n\}, \ldots, \{n^{m_2-1}\alpha^n\}; \ldots; \{\alpha_k^n\}, \{n\alpha_k^n\}, \ldots, \{n^{m_k-1}\alpha_k^n\}$$

are linearly independent (try to show this).

Since the space S of solutions is r dimensional, the above list is a basis for S, and all solutions have to be linear combinations of these.

The complete statement is given in the next page.

Notational change From now on, a solution $\{f_n\}$ of $(*)$ will be written as $f_n, n \geq 0$ (or sometimes, just f_n) instead of the more cumbersome $\{f_n\}$.

It will be clear from the context if f_n means the entire sequence $\{f_n\}$ or just an individual member of the sequence.

Theorem (a) Let α be a root with multiplicity m of the characteristic equation $x^r - c_1x^{r-1} - c_2x^{r-2} - \dots - c_{r-1}x - c_r = 0$ associated with the linear homogeneous recurrence relation (*). Then the m sequences

$$a_n = \alpha^n,\ n\alpha^n,\ \dots,\ n^{m-1}\alpha^n$$ ← (We are abandoning the curly brackets { } and write α^n for $\{\alpha^n\}$)

are linearly independent.

(b) Let $\alpha_1, \alpha_2, \dots, \alpha_k$ be distinct non-zero roots with multiplicities $m_1, m_2, \dots, m_k$ respectively for the characteristic function associated with (*). Then

$$\begin{aligned} a_n = {} & \alpha_1^n,\ n\alpha_1^n,\ \dots,\ n^{m_1-1}\alpha_1^n; \\ & \alpha_2^n,\ n\alpha_2^n,\ \dots,\ n^{m_2-1}\alpha_2^n; \qquad n \geq 0 \\ & \vdots \qquad\quad \vdots \qquad\qquad \vdots \\ & \alpha_k^n,\ n\alpha_k^n,\ \dots,\ n^{m_k-1}\alpha_k^n \end{aligned}$$

are linearly independent solutions of (*). Their linear combinations form the general solution of the recurrence relation (*).

Examples

1. Solve $a_n = 2a_{n-1} + 3a_{n-2}$ with $a_0 = a_1 = 1$.

Soln: The characteristic equ is

$$x^2 - 2x - 3 = 0$$

i.e., $$(x-3)(x+1) = 0$$

i.e. $$x = 3 \text{ or } x = -1.$$

Therefore the general solution is

$$a_n = c\,3^n + d\,(-1)^n$$

where c and d are constants. We are given that $a_0 = a_1 = 1$. Setting $n=0$ we get

$$a_0 = c + d, \text{ i.e., } \quad c+d = 1$$

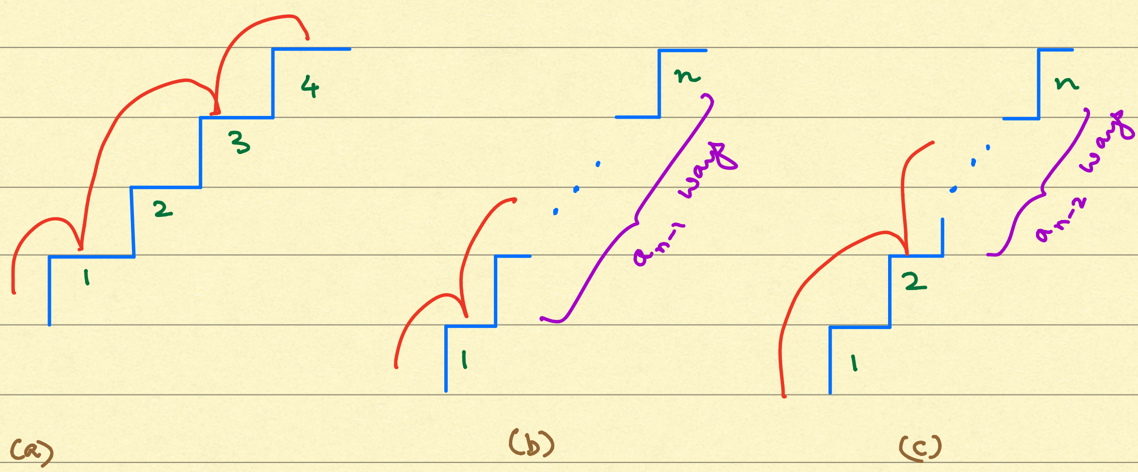
and setting $n=1$ we get

$$a_1 = 3c - d, \text{ i.e., } \quad 3c - d = 1$$

It follows that $c = \frac{1}{2}$ and $d = 1/2$. Thus

$$a_n = \frac{(-1)^n + 3^n}{2} \quad \longleftarrow \text{Answer}$$

2. Climbing stairs: An elf has a staircase of n stairs to climb. Each step it can cover either one stair or two stairs. Let a_n be the number of different ways for the elf to ascend the n-stair staircase.



It is easy to see that $a_1 = 1$, $a_2 = 2$ (why?), and $a_3 = 3$. For $n = 4$, figure (a) shows one way (steps of size 1, 2, and 1). There are actually a total of five ways of climbing, so

$a_4 = 5$. Figure (a) is $1+2+1$ (first one stair, then two, then one). Other four ways are $1+1+2$, $2+1+1$, $2+2$, $1+1+1+1$ with obvious meaning attached to these sums.

Look at pictures (b) and (c). If the elf's first step is 1 stair, then the elf has a_{n-1} ways to get from the first stair to the top. If the elf's first step is of size 2, then the elf has a_{n-2} ways to go from there to the top. It follows that

$$a_n = a_{n-1} + a_{n-2} \qquad n \geq 3$$

The characteristic eqn is

$$x^2 - x - 1 = 0$$

From the quadratic formula, the roots are

$$x = \frac{1+\sqrt{5}}{2} \text{ and } \frac{1-\sqrt{5}}{2}.$$

Thus the general solution is

$$a_n = c\left(\frac{1+\sqrt{5}}{2}\right)^n + d\left(\frac{1-\sqrt{5}}{2}\right)^n$$

where c, d are constants. However we know $a_1 = 1$ and $a_2 = 2$. Plugging in these values we get $c = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)$ and $d = \frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)$.

3. $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}$; $n \geq 3$

Solution:

Here $r = 3$ and the characteristic eqn. is

$$x^3 - 7x^2 + 16x - 12 = 0$$

i.e., $$(x-2)^2(x-3) = 0$$

So $\qquad x = \underbrace{2,2}_{\text{repeated root of multiplicity 2.}}, 3$

Since the root has multiplicity 2 and the root 3 has multiplicity 1, we see that the general soln is

$$a_n = \alpha \cdot 2^n + \beta n 2^n + \gamma 3^n, \; n \geq 0$$ ← Answer

where α, β, γ are constants.

4. In the above recurrence relation if $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, what is a_n?

<u>Soln</u>: From the general soln $a_n = \alpha 2^n + \beta n 2^n + \gamma 3^n$ we see that (setting $n = 0, 1,$ and 2)

$$a_0 = \alpha + \gamma, \quad a_1 = 2\alpha + 2\beta + 3\gamma, \quad a_2 = 4\alpha + 8\beta + 9\gamma.$$

Thus we have

$$\left.\begin{aligned} \alpha + \quad\;\; \gamma &= 0 \\ 2\alpha + 2\beta + 3\gamma &= 1 \\ 4\alpha + 8\beta + 9\gamma &= 2 \end{aligned}\right\} (+)$$

It is easy to see that $\alpha = 2$, $\beta = \frac{3}{2}$, $\gamma = -2$ is the solution of $(+)$. Therefore

$$a_n = 2(2^n) + \frac{3}{2}(n 2^n) - 2(3^n), \qquad n \geq 0.$$

This can be simplified to

$$a_n = 2^{n-1}(4 + 3n) - 2(3^n).$$ ← Answer

Section 7.4 Solution of inhomogeneous recurrence relations

An inhomogeneous linear recurrence relation looks like

The book does not use the term linear, so we will also drop the term.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_{r-1} a_{n-r+1} + c_r a_{n-r} + f(n)$$

We have already done one example — the recurrence relation for the number of regions a_n that n lines in a plane (none parallel, no three intersecting at a point) divide the plane into is

$$a_n = a_{n-1} + n.$$

this makes it inhomogeneous

$f(n) = n$.

And the solution was

$$a_n = \frac{1}{2} n(n+1).$$

General and particular solutions: Consider the inhomog. recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_r a_{n-r} + f(n)$, $n \geq r$. The homogeneous part of this recurrence rel'n is defined to be $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_r a_{n-r}$, $n \geq r$. A particular solution to the inhomogeneous rel'n is any soln $p(n)$ of the inhomog. rel'n. It is easy to see that if $A_1 \omega_{1,n} + A_2 \omega_{2,n} + \dots + A_r \omega_{r,n}$ is a general soln of the homogeneous part then the general solution of the inhomogeneous relation is $A_1 \omega_{1,n} + \dots + A_r \omega_{r,n} + p(n)$. This is because any two solutions of the inhomogeneous eqn differ by a solution of the homog. part and vice versa.

How does one find particular solutions?

Here is the statement when $r=1$, and $f(n)$ a polynomial

Theorem: Let

$$a_n = c a_{n-1} + f(n), \quad n \geq 1 \quad \text{———} \quad (1)$$

be an inhomogeneous recurrence relation with
$f(n) = \sum_{i=1}^{k} B_i n^i$ (B_i's constants).

(a) If $c \neq 1$, then (1) has a particular soln. of the form

$$p(n) = d_0 + d_1 n + d_2 n^2 + \dots + d_k n^k$$

where the coefficients $d_0, d_1, \dots, d_k$ are recursively determined via

$$d_k = \frac{B_k}{1-c}$$

$$d_i = \frac{1}{1-c}\left[B_i + A \sum_{j=i+1}^{k} (-1)^{j-i} \binom{j}{i} A_j\right],$$

$$0 \leq i \leq k-1.$$

(b) If $c=1$, then (1) has a particular solution

$$p(n) = a_0 + \sum_{i=1}^{n} f(i)$$

Remark: The table given at the bottom of p.305 of the textbook is more useful than the Theorem above. We reproduce it below (the table is only given for $c \neq 1$).
Note that $c=1$, then we have $a_1 - a_0 = f(1)$, $a_2 - a_1 = f(2)$, $\dots$, $a_n - a_{n-1} = f(n)$, and hence upon adding these we get
$a_n - a_0 = f(1) + f(2) + \dots + f(n)$, i.e. $a_n = a_0 + \sum_{i=1}^{n} f(i)$

$f(n)$	Particular soln $p(n)$
d (a constant)	B
dn	$B_1 n + B_0$
dn^2	$B_2 n^2 + B_1 n + B_0$
$e\, d^n$	$B d^n$. ← This case is not mentioned in the Theorem but is very important.

Table for $c \neq 1$ (see p. 305 of text book)

<u>Example</u> : Solve $a_n = 3a_{n-1} - 4n$, $n \geq 1$; .

<u>Soln</u> : The general soln of the homogeneous part is $a_n^* = 3^n c$. Let $p(n) = B_1 n + B_0$ be a particular solution. Then

$$\left.\begin{aligned} B_1 n + B_0 &= 3\left[B_1(n-1) + B_0\right] - 4n \\ &= 3B_1 n + 3(B_0 - B_1) - 4n \\ &= (3B_1 - 4)n + 3(B_0 - B_1). \end{aligned}\right\} \quad n \geq 0$$

Comparing coefficients we get

$$B_1 = 3B_1 - 4 \quad \text{and} \quad B_0 = 3B_0 - 3B_1$$

This means $B_1 = 2$ and $B_0 = 3$. Thus the general solution is

$$\boxed{a_n = 2n + 3 + c3^n, \quad n \geq 1.} \leftarrow \text{Answer}$$

If a_0 is given, then c can be worked out for $a_0 = 3 + c$. For example, if $a_0 = 1$, then $c = -2$ and $a_n = 2n + 3 - 2(3^n)$. If $a_0 = 0$, then $c = -3$ and $a_n = 2n + 3 - 3^{n+1}$.

Section 7.5 Solutions with generating functions:

Example: Consider the Fibonacci relation

$$a_n = a_{n-1} + a_{n-2}, \quad n \geq 3$$

with $a_1 = 1$ and $a_2 = 2$. The conditions $a_1 = 1$, $a_2 = 2$ are equivalent to $a_0 = 1$, $a_1 = 1$ and

$$a_n = a_{n-1} + a_{n-2}, \quad n \geq 2. \qquad \text{——} \quad (*)$$

Let

$$g(x) = \sum_{n=0}^{\infty} a_n x^n \quad \longleftarrow \text{generating fcn for } a_n.$$

From $(*)$ we have

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n$$

i.e.

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{m=1}^{\infty} a_m x^{m+1} + \sum_{r=0}^{\infty} a_r x^{r+2}$$

$$(m = n-1, \quad r = n-2)$$

$$\Rightarrow \quad \sum_{n=2}^{\infty} a_n x^n = x \sum_{m=1}^{\infty} a_m x^m + x^2 \sum_{r=0}^{\infty} a_r x^r$$

$$\Rightarrow \quad \left(\sum_{n=0}^{\infty} a_n x^n\right) - a_0 - a_1 x = x\left[\left(\sum_{m=0}^{\infty} a_m x^m\right) - a_0\right]$$

$$+ x^2 \sum_{r=0}^{\infty} a_r x^r.$$

This means

$$g(x) - a_0 - a_1 x = x\left(g(x) - a_0\right) + x^2 g(x)$$

Since $a_0 = a_1 = 1$, the above gives

$$g(x) - 1 - x = x\,(g(x) - 1) + x^2 g(x)$$

This is a functional equation.

Thus

$$(1 - x - x^2)\, g(x) = 1 + x - x = 1$$

i.e.

$$g(x) = \frac{1}{1 - x - x^2}.$$

This is what is really the functional equ.

Note: The denominator $1 - x - x^2$ is related to the char. equ $x^2 - x - 1 = 0$. In gen'l if $x^r + c_1 x^{r-1} + \dots + c_r = 0$ is the char. equ, then $g(x)$ will have $1 + c_1 x + c_2 x^2 + \dots + c_r x^r$ as its denominator. The numerator will depend upon initial conditions.

Use partial fractions and get

$$g(x) = \frac{\alpha_1/\sqrt{5}}{1 - \alpha_1 x} - \frac{\alpha_2/\sqrt{5}}{1 - \alpha_2 x}$$

where $\alpha_1 = \frac{1}{2}(1 + \sqrt{5})$ and $\alpha_2 = \frac{1}{2}(1 - \sqrt{5})$

Now

$$\frac{\alpha_1}{\sqrt{5}} \cdot \frac{1}{1 - \alpha_1 x} = \frac{\alpha_1}{\sqrt{5}} \left\{ 1 + (\alpha_1 x) + (\alpha_1 x)^2 + (\alpha_1 x)^3 + \dots \right\}$$

$$= \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \alpha_1^{n+1} x^n$$

and

$$\frac{\alpha_2}{\sqrt{5}} \cdot \frac{1}{1 - \alpha_2 x} = \frac{\alpha_2}{\sqrt{5}} \left\{ 1 + (\alpha_2 x) + (\alpha_2 x)^2 + (\alpha_2 x)^3 + \dots \right\}$$

$$= \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \alpha_2^{n+1} x^n.$$

It follows that

$$g(x) = \sum_{n=0}^{\infty} \frac{(\alpha_1^{n+1} - \alpha_2^{n+1})}{\sqrt{5}} x^n$$

Thus

$$a_n = \frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\sqrt{5}}$$

i.e.

$$\boxed{a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.}$$

Note: The above is not a very useful formula for computing a_n. It is easier to compute a_{10} by recursively computing $a_2, a_3, \ldots,$ up to a_{10} than setting $n=10$ in the above formula.

Example:

1. Find a functional eqn for the generating function associated to

$$a_n = 4a_{n-1} + 2a_{n-2} - 3, \quad a_0 = a_1 = 1.$$

Soln:

Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$.

From the recurrence relation we have

$$\sum_{n=2}^{\infty} a_n x^n = 4\sum_{n=2}^{\infty} a_{n-1} x^n + 2\sum_{n=2}^{\infty} a_{n-2} x^n$$

This means

$$g(x) - a_0 - a_1 x = 4x\,(g(x) - a_0) + 2x^2 g(x)$$

The above is a functional eqn for $g(x)$. But let us simplify

$$(1 - 4x - 2x^2)\,g(x) = a_0 + (a_1 - 4a_0)x$$
$$= 1 - 3x$$

Thus $$g(x) = \frac{1-3x}{1-4x-2x^2}.$$

<u>Example</u>: Find functional eqn for the generating function associated to

$$a_n = \sum_{i=0}^{n-1} a_i a_{n-1-i} \quad (n \geq 1), \quad a_0 = 1.$$

<u>Solution</u>:

Let $$g(x) = \sum_{n=0}^{\infty} a_n x^n$$

From the recurrence relation (note $n \geq 1$) we have

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \left\{ \sum_{i=0}^{n-1} a_i a_{n-1-i} \right\} x^n$$

So $$g(x) - a_0 = \sum_{n=1}^{\infty} \left\{ \sum_{i=0}^{n-1} a_i x^i a_{n-1-i} x^{n-1-i} \right\} x$$

Let $m = n-1$. Then above becomes

$$g(x) - a_0 = \sum_{m=0}^{\infty} \left\{ \sum_{i=0}^{m} a_i x^i a_{m-i} x^{m-i} \right\} x$$

$$= x \left(\sum_{m=0}^{\infty} a_m x^m \right) \left(\sum_{m=0} a_m x^m \right)$$

$$= x\, g(x)^2$$

Thus

$$g(x) - 1 = x\, g(x)^2$$

$$x g(x)^2 - g(x) + 1 = 0 \longleftarrow \text{Functional eqn.}$$

This means

$$g(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Note that $a_0 = g(0)$, and $a_0 = 1$. This means $g(0) = 1$. This forces

$$g(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

It is well-known that

$$\sqrt{1-4x} = 1 - 2 \sum_{n=0}^{\infty} \frac{(2n)!}{n!\,(n+1)!}\, x^{n+1}. \qquad \text{—— } (**)$$

Check how this follows from this

From this one concludes that

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n.$$

Thus

$$\boxed{a_n = \frac{1}{n+1}\binom{2n}{n}} \longleftarrow \text{Catalan numbers.}$$

The expansion $(1-4x)^{1/2} = 1 - 2\sum_{n=0}^{\infty} \frac{(2n)!}{n!\,(n+1)!} x^{n+1}$ has to do with Newton's binomial theorem for fractional exponents (which is a special case of MacLaurin's series or Taylor's series). We review that briefly next.

The Binomial Theorem for fractional exponents

The trick used here is Taylor's series/McLaurin's series:

Set $y = 4x$ in this to get the expansion for $(1-4x)^{1/2}$.

$$(1+y)^{1/2} = 1 + \frac{1}{2}y + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}y^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}y^3 + \dots + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-3}{2}\right)}{n!}y^n + \dots$$

The expression $\frac{1}{2}\left(-\frac{1}{2}\right)\left(-3/2\right)\cdots\left(-\frac{2n-3}{2}\right)$ can be re-written as

$$\frac{1}{2}\left(-\frac{1}{2}\right)\left(-3/2\right)\cdots\left(-\frac{2n-3}{2}\right) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(n-1)\right)$$

Compare

$$\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(n-1)\right)}{n!}$$

with

$$\binom{r}{n} = \frac{r(r-1)\cdots(r-(n-1))}{n!}$$

Therefore

$$\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(n-1)\right)}{n!}$$

can be regarded as

$$\binom{1/2}{n}$$

and the Maclaurin/Taylor expansion for $(1+y)^{1/2}$ as

$$(1+y)^{1/2} = \sum_{n\geq 0}\binom{1/2}{n}y^n$$

This was in fact discovered by Newton and part of his "binomial theorem for fractional exponents".