March 22, 2018

# Lecture 20

One of the problems we examined last time was the problem of distributing $r$ distinct apples amongst 5 people such that each person gets at least one apple. If $a_r$ is the # of ways of distributing according to the above rules, we found that the exponential generating funtion for the problem is

$$g(x) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)^5$$

and this could be re-written as $g(x) = (e^x - 1)^5$. Using this we saw

$$(*) \quad a_r = \begin{cases} 0 & \text{if } r = 0 \\ 5^r - 5(4^r) + 10(3^r) - 10(2^r) + 5 & \text{if } r \geq 1 \end{cases}$$

Note that $a_r = 0$ is not a surprise. The number of ways of distributing 0 apples amongst 5 persons so that each one gets one apple is zero. The same combinatorial argument shows $a_0 = a_1 = a_2 = a_3 = a_4 = 0$. We can check this for our formula directly. For example, when $r = 1$, $a_1 = 5 - 5(4) + 10(3) - 10(2) + 5$ $= 5 - 20 + 30 - 20 + 5 = 0$.

The fact that $a_0 = a_1 = a_2 = a_3 = a_4 = 0$ can be seen in another way. Recall that $a_r = r!\, c_r$, where $c_r$ is the coefficient of $x^r$ in $g(x)$. But $g(x) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)^5 = \left[x\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots\right)\right]^5$

i.e., $$g(x) = x^5\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots\right)^5$$
$$= x^5 + c_6 x^6 + c_7 x^7 + \dots$$

It follows that $c_0 = c_1 = c_2 = c_3 = c_4 = 0$, i.e., $a_0 = a_1 = a_2 = a_3 = a_4 = a_5 = 0$.

Similarly, by combinatorics, if $r=5$, there are $5!=120$ ways of distributing 5 distinct apples amongst 5 people. Thus

$$a_5 = 120$$

Another way: Since $g(x) = x^5 + c_6 x^6 + c_7 x^7 + \cdots$ (see above), we have $c_5 = 1$, which means

$$a_5 = 5!\,(1) = 120.$$

Let us verify this using formula (*), i.e.

$$a_r = 5^r - 5(4^r) + 10(3^r) - 10(2^r) + 5 \quad \text{if } r \geq 1$$

This yields

$$\begin{aligned} a_5 &= 5^5 - 5(4^5) + 10(3^5) - 10(2^5) + 5 \\ &= 3125 - 5120 + 2430 - 320 + 5 \\ &= 5560 - 5440 \qquad (3125+2430+5=5560,\ 5120+320=5440) \\ &= 120. \end{aligned}$$

Thus, the three different calculations agree.

Let us give yet another way of looking at the formula

$$a_r = 5^r - 5(4^r) + 10(3^r) - 10(2^r) + 5, \quad r \geq 1.$$

If we had no condition imposed upon us and were allowed to distribute the $r$ distinct apples amongst 5 persons freely, then we could do this in $5^r$ ways. From these $5^r$ ways let us subtract the distributions which give no apples to somebody. There are 5 ways of picking a person who does not get an apple. Pick one. We distribute the $r$ apples freely amongst the remaining 4 persons. There are $4^r$ ways of doing this — hence $5(4^r)$ ways of have been accounted for. But there has been a double count. We have conted the cases of where more than one person has received no apples twice. We have to add the counts for these cases back. So pick two people who don't get apples. There are $\binom{5}{2} = 10$ ways of doing this. Freely distribute the r apples amongst the remaining three persons. There are $3^r$ ways of doing this. This accounts $10(3^r)$ ways. But there is a double count again — the cases where three or more persons have not got an apple have been conted twice. The situation so far is

$$a_r = 5^r - (\#\text{ of cases where someone gets no apples})$$
$$= 5^r - 5(4^r) + (\#\text{ of cases where two or more persons get no apples})$$
$$= 5^r - 5(4^r) + 10(3^r) + (\#\text{ cases where 3 or more persons get no apples})$$

The number in parenthesis in the last line is found by selecting three persons who don't get apples and distributing apples amongst the rest and subtracting a double count. So

$$a_r = 5^r - 5(4^r) + 10(3^r) - \binom{5}{3}(2^r) + (\#\text{ of cases where 4 or more persons get no apples})$$
$$= 5^r - 5(4^r) + 10(3^r) - 10(2^r) + (\#\text{ of cases where 4 or more persons get no apples})$$

Now $r \geq 1$. So it is not possible for all five persons to get no apples. Choose four people to give no apples to (there are $\binom{5}{4} = 5$ ways of doing this) and give all the r apples to the lucky remaining person. Thus, if $r \geq 1$

$$a_r = 5^r - 5(4^r) + 10(3^r) - 10(2^r) + 5$$

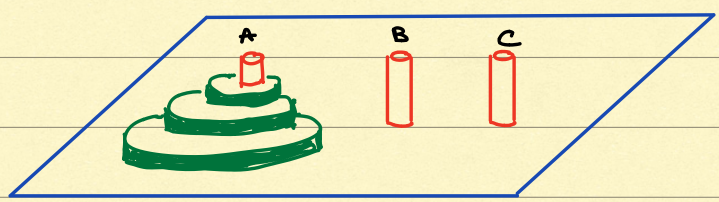
The method of counting described just now is called the inclusion - exclusion principle and is very important.

# Chapter 7 Recurrence relations

## Section 7.1

Tower of Hanoi : Suppose we have $n$ discs all of different sizes and three pegs, A, B, and C, with the discs pegged to A and arranged in a descending order of size, the largest at the bottom and the smallest at the top. (See picture below for $n=3$). We are allowed to move the discs to different pegs with one caveat —— no disc can be put on top of a smaller disc. Let

$a_n$ = # of moves required to move $n$ discs from A to C following the above rule.




Initially all discs on the first peg, i.e., the left-most peg. The largest at the bottom, the smallest at the top.

We could move the top $n-1$ discs to B using $a_{n-1}$ moves. The bottom most disc can be now moved to C. We could now move the $n-1$ discs on B to C, and this requires another $a_{n-1}$ number of moves. This gives the formula

$$a_n = 2a_{n-1} + 1.$$ ← Example of a recurrence relation.

The relation

$$a_n = 2a_{n-1} + 1$$

is an example of a recurrence relation. Clearly if we know $a_1$, we can work out $a_2$, and then $a_3$, and so on and work out $a_n$. Now clearly $a_1 = 1$. It follows that $a_2 = 2(1)+1 = 3$, $a_3 = 2(3)+1 = 7$, $a_4 = 2(7)+1 = 15 \ldots$.
One can prove, using induction, that $a_n = 2^n - 1$, $n \geq 1$.
Indeed if the formula is true for $n=k$, then
$a_{k+1} = 2(2^k - 1) + 1 = 2^{k+1} - 1$, giving the proof by induction (we can see easily that the formula is true for $k=1$).