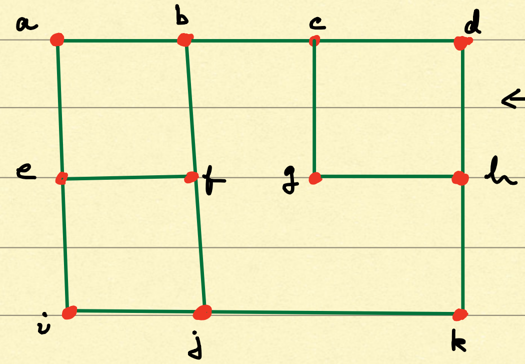
Jan 9, 2018 Lecture 2



Last time we were looking at (Example 3). Let us use this to introduce models for certain phenomenon.

(a) Network security: Suppose the figure above represents a telephone network; the vertices representing switching centres, and the edges telephone lines.

Question: Which set of lines and switching centres must stay in service to avoid disconnecting the network?

(No unique solution!)

Can remove any telephone line, or switching centre, and the network is still fully "connected" (will give a precise definition of a connected graph later in this lecture).

If (a, b) and (a, e) are removed then it is impossible to "reach" the vertex a. In other words if the telephone lines (a, b) and (a, e) are removed, then the switching centre a is isolated.

If (b, c) and (j, k) are removed then again the network is "disconnected". For example there is no set of telephone lines which connects e to c once we remove (b, c) and (j, k).

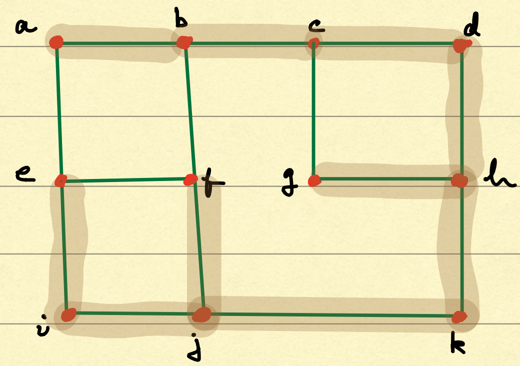
Exercise: Find all "disconnecting sets" consisting of two edges in the figure above.

Here is a different question: Find a minimal set of edges needed to link all the 11 vertices. Minimal means if any of the edges is removed from the set, then not all switching centres are connected.

Again there is no unique answer. Check that

$\{(a,b), (b,c), (c,d), (d,h), (h,g), (h,k), (k,j), (j,f), (j,i), (i,e)\}$

is an answer.



Some definitions:

If (a,b) is an edge of a graph $G = (V,E)$ then the vertices a and b are said to be adjacent. In this case we say that the edge (a,b) is incident to a and b.

A path P is a sequence of distinct vertices, written

$P = x_1 - x_2 - x_3 - \dots - x_n$

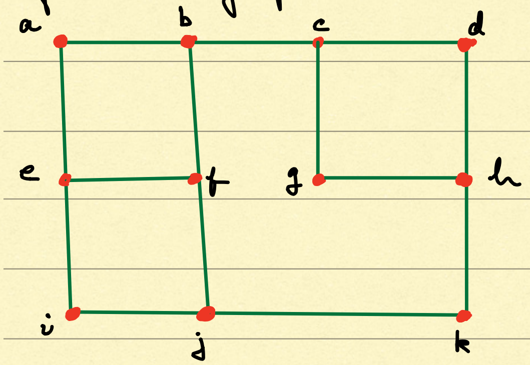
with each pair of consecutive vertices being an edge.

If in addition there is an edge (x_n, x_1) the sequence is a circuit and is written $x_1 - x_2 - \dots - x_n - x_1$.

A graph is connected if there is a path between every pair of vertices. Otherwise it is disconnected.

A set C of vertices of G is called an edge cover if every edge is incident to at least one vertex in C.

Street surveillance: Let us go back to our favourite graph:



Suppose it represents a section of Toronto's street map. Suppose the edges represent blocks and the vertices corners. We'd like to station police officers at corners (not necessarily all corners) so that every block has an officer at one of its corners. To save resources, we'd like to do this with a minimum number of officers. Here is how we can go about the problem.

There are 14 edges.

A vertex is incident to at most 3 edges in our graph. (This means that the degree of any vertex in our graph is at most 3 — we'll define degree after this example). If officers are stationed at 4 vertices, then the most number of blocks they can cover is $4 \times 3 = 12$ blocks. But we have 14 blocks.

So 4 officers are never going to be enough!

How about 5 officers? Will they be enough?

Note that the vertices b, c, e, f, h, and j all have 3 edges incident upon them (this means, they have degree 3). There are 6 vertices. Suppose we station police officers on five of these. Then these officers will cover 5×3 blocks, i.e, 15 blocks. There are only 14 blocks, and so one block will have an officer on either end.

Question: Suppose we place officers at b, f, j, c, and h, will we have covered all the blocks?

Clearly not! The edges (a,e) and (e,i) will have no officers at the vertices they are incident to.

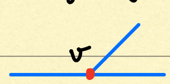
Q: What about officers at c, h, j, e, and b? Will that work?

Answer: Yes.

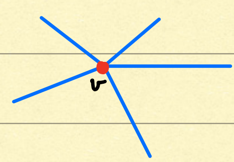
Q: Is that the only solution with 5 officers?

A: Yes. Read the argument from the book. If you have problems understanding the argument use help (recitation, office hours etc).

Definition: Let $G = (V, E)$ be a graph and $v \in V$ a vertex. The degree of v is the number of edges of G incident to v.



degree $v = 3$



degree $v = 5$.

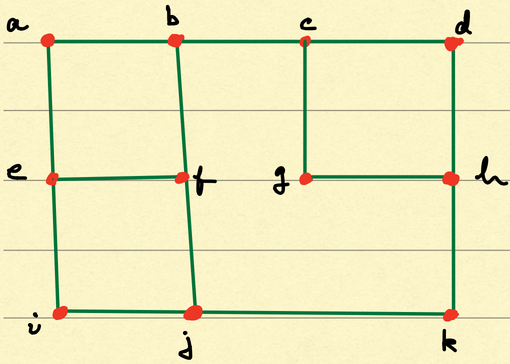
Definition: Let $G = (V, E)$ be a graph. A set of vertices without an edge between any two of them is called an independent set of vertices.

In the example we discussed $\{a, f, g, k\}$ is an independent set of vertices. Of course they could be many others ($\{j, c\}$ for example).

Scheduling Meetings: The provincial legislature for Ontario (at Queen's Park, right near Gerstein library) has a number of committees. An MPP could be a member of many different committees. If an MPP is a member of committee a and of committee b, then a and b cannot meet at the same time. We can draw a graph to represent the situation, with vertices standing for committees and drawing an edge between two vertices if the two committees have a common member.

Note: A set of committees can schedule a meeting at the same time (in different rooms in Queen's Park, of course!) if and only if the corresponding set of vertices is independent!

Suppose, once again, the graph for this situation is the one we have become familiar with:



How many committees can meet at one time?

In other words: Same question

What is the largest independent set in the graph?

In general it is very hard to find the largest independent set in a graph.

For the above graph one can show that

$$\{a, d, f, g, i, k\}$$

is an independent set, and that all other independent sets have fewer elements.

Trees: There are special graphs called trees. Trees can be characterized as graphs which are connected and which have a unique path between any pair of vertices.

This was NOT done in class. Please read!

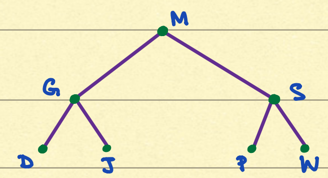
Spelling Checker: Spelling checkers look at each "word" in a document and tries to match

(word = binary number)

it with some word in its "dictionary", which is just a list of binary numbers stored somewhere. Typically dictionaries have 100,000 words.

<u>Simplified problem</u>: Say our words are one letter, and our "dictionary" is the English alphabet (26 letters) and our spell-checker can decide, given two letters X & Y, if $X \le Y$ or not.

Suppose we have an unknown letter X.



Situation represented by this graph.

I. Test if $X \le M$ or not.
Either
I_1 : $X \le M$
or
I_2 : $X > M$.

II. If I_1, test if $X \le G$ or not.
If I_2, test if $X \le S$ or not.
In I_1 have 2 possibilities:
I_{11} : $X \le G$
or I_{12} : $X > G$
In case I_2, have
I_{21} : $X \le S$
or
I_{22} : $X > S$

III. I_{11}: Test $X \le D$ or not
I_{12}: Test $X \le J$ or not
I_{21}: Test $X \le P$ or not
I_{22}: Test $X \le W$ or not.

Note that it is a directed graph. And is a tree.

A word processor would use a similar but larger tree of comparisons.

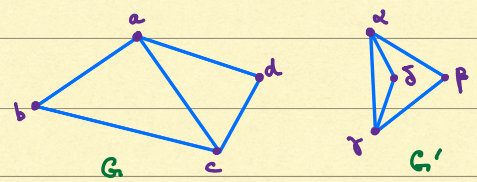
If the dictionary has 100,000 words, and X is an unknown word, then in just 12 rounds of comparisons, one can reduce the problem to comparing X against a list of just 25 words.

Chapter 3 is where trees are studied.

<u>Read</u> : Example 6 on pages 9 & 10 from the book.

§1.2 Isomorphisms

How different are the following graphs?



Really the graphs are $G = (V, E)$, $G' = (V', E')$ with

$V = \{a, b, c, d\}$, $E = \{(a,b), (a,c), (a,d), (b,c), (c,d)\}$

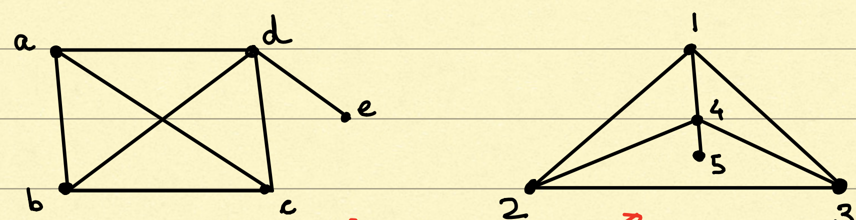
and

$V' = \{\alpha, \beta, \gamma, \delta\}$, $E' = \{(\alpha,\beta), (\alpha,\gamma), (\alpha,\delta), (\beta,\gamma), (\beta,\delta)\}$

In some sense they are the same graphs!

Definition: Two graphs G and G' are called isomorphic if there exists a one-to-one correspondence between the vertices in G and the vertices in G' such that a pair of vertices are adjacent in G if and only if the corresponding pair of vertices are adjacent in G'.

Example:



These are isomorphic!

$$\left.\begin{array}{l} a \longleftrightarrow 1 \\ b \longleftrightarrow 2 \\ c \longleftrightarrow 3 \\ d \longleftrightarrow 4 \\ e \longleftrightarrow 5 \end{array}\right\}$$

Check that this gives the isomorphism (explained in class).

Notation: Let v be a vertex of a graph G.

$\deg v$ = degree of v.

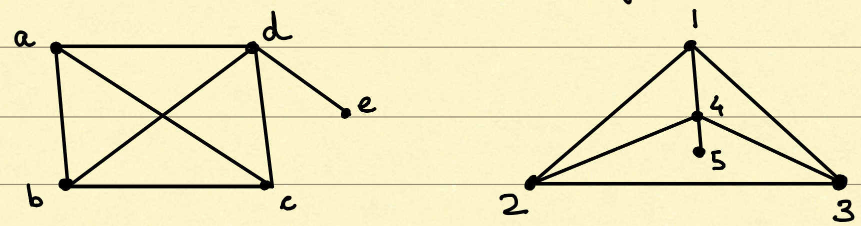
notation for degree.

Subgraphs: If $G = (V, E)$ is a graph, then a subgraph G' of G is a graph whose vertices are a subset of V and edges are a subset of E. This means $G' = (V', E')$ is a subgraph of G if it is a graph such that $V' \subseteq V$ and $E' \subseteq E$.

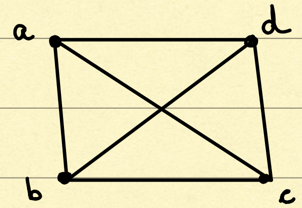
The same definition also works for a directed graph.

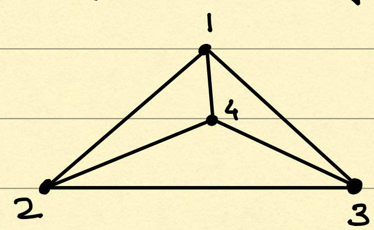
Remark: If G and H are isomorphic graphs then a subgraph G' of G gives rise to an isomorphic subgraph H' of H whose vertices and edges are the ones corresponding to the vertices and edges of G'.

Consider again the two graphs.



Remove vertices e and 5 (and their adjacent edges).





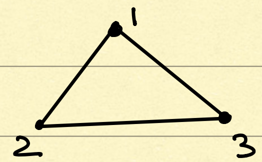
By moving 4 in a north-easterly direction past the edge (1,3) it is easy to see that the two subgraphs are isomorphic. From here to showing that the whole graphs are isomorphic is easy.

Complete graph on n vertices K_n:

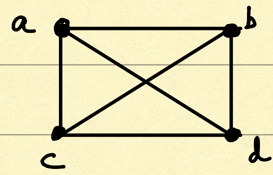
A graph with n vertices in which each vertex is adjacent to all other vertices is called a complete graph on n vertices, and is denoted K_n.

K_2

K_3



K_4



Defn : A graph $G = (V, E)$ is said to be complete if every pair of distinct vertices in G are adjacent.

Note: If G is a complete graph and has n vertices then G is isomorphic to K_n.