March 20, 2018

Lesson 19

Section 6.2 Calculating Coefficients of Generating Functions

Let us prove

$$\boxed{\frac{1}{(1-x)^n} = 1 + \binom{1+n-1}{1}x + \binom{2+n-1}{2}x^2 + \dots + \binom{r+n-1}{r}x^r + \dots}$$

Since $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$, therefore

$$\frac{1}{(1-x)^n} = (1 + x + x^2 + \dots)^n$$

Formal products are of the form $x^{e_1}x^{e_2}\dots x^{e_n}$ and the coefficient of x^r is the number of integer solutions of $e_1 + e_2 + \dots + e_n = r$ with $e_i \geq 0$. The answer, as we know from Chapter 5, is $\binom{n+r-1}{r}$. That proves the identity.

Examples

1. Find the coefficient of x^{20} in $(x^3 + x^4 + x^5 + \dots)^6$.

Soln:

$$\begin{aligned}(x^3 + x^4 + x^5 + \dots)^6 &= [x^3(1 + x + x^2 + \dots)]^6 \\ &= x^{18}(1 + x + x^2 + \dots)^6 \\ &= \frac{x^{18}}{(1-x)^6}\end{aligned}$$

Coeff of x^{20} in $\frac{x^{18}}{(1-x)^6}$ = coeff of x^2 in $(1-x)^{-6}$

From what we proved earlier, the coefficient of x^2 in the power series expansion of $(1-x)^{-6}$ is

$$\binom{2+6-1}{2} = \binom{7}{2} \longleftarrow \text{Answer}$$

//

Before we do the next example we give the following formula (see formula (6) in the display on p.257 of the book).

$$\boxed{\begin{array}{l} \text{If } h(x) = f(x)\,g(x), \text{ where } f(x) = a_0 + a_1 x + a_2 x^2 + \dots \text{ and} \\ g(x) = b_0 + b_1 x + b_2 x^2 + \dots, \text{ then} \\ h(x) = a_0 b_0 + (a_1 b_0 + a_0 b_1) x + (a_2 b_0 + a_1 b_1 + a_0 b_2) x^2 + \dots \\ \qquad + (a_r b_0 + a_{r-1} b_1 + a_{r-2} b_2 + \dots + a_0 b_r) + x^r + \dots \end{array}}$$

2. Use generating functions to find the number of ways to collect \$15 from 20 distinct people if each of the first 19 can give a dollar (or nothing) and the twentieth person can give either \$1 or \$5 (or nothing).

Solution: We have to find the number of integer solutions of

$$e_1 + e_2 + \dots + e_{19} + e_{20} = 15 \qquad 0 \le e_i \le 1 \text{ for } 1 \le i \le 19$$
$$\text{and } e_{20} \in \{0, 1, 5\}.$$

This is equivalent to finding the coefficient of x^{15} in

$$(1+x)^{19}(1+x+x^5).$$

Here is how this is done:

$$(1+x)^{19} = \sum_{r=0}^{19} \binom{19}{r} x^r \qquad \text{(Binomial Theorem)}$$

According to the formula we displayed above for the coefficients of the power series obtained as the product of two power series,

Coeff. of x^{15} in $(1+x)^{19} \cdot (1+x+x^5)$

$$= \text{Coeff. of } x^{15} \text{ in } \left[\left(\sum_{r=0}^{19} \binom{19}{r} x^r\right)\right](1+x+x^5)$$

$$= \binom{19}{15}\cdot 1 + \binom{19}{14}\cdot 1 + \binom{19}{13}\cdot 0 + \binom{19}{12}\cdot 0 + \binom{19}{11}\cdot 0 + \binom{19}{10}\cdot 1 + 0$$

$$= \binom{19}{15} + \binom{19}{14} + \binom{19}{10} \longleftarrow \text{Answer.}$$

//

Here is another well known formula:

$$\boxed{1+x+x^2+\dots+x^m = \frac{1-x^{m+1}}{1-x}}$$

To prove this, note that

$$(1-x)(1+x+x^2+\dots+x^m) = (1+x+x^2+\dots+x^m) - x(1+x+x^2+\dots+x^{m-1}+x^m)$$

$$= (1+x+x^2+\dots+x^m) - (x+x^2+\dots+x^m+x^{m+1})$$

$$= 1-x^{m+1}.$$

The result follows. //

3. Use generating functions to find the number of ways to distribute r gym bags 12 athletes if

(a) Each athlete gets at least one bag

(b) Each athlete gets an even number of bags.

Soln

For (a) the generating function is $(x+x^2+x^3+\dots)^{12}$. We have to find the coefficient of x^r.

$$(x+x^2+x^3+\dots)^{12} = x^{12}(1+x+x^2+\dots)^{12}$$
$$= x^{12}\sum_{s=0}^{\infty}\binom{12+s-1}{s}x^s$$

The coefficient of x^r in $(x+x^2+\dots)^{12}$ is therefore the coefficient of x^{r-12} in $\sum_{s=0}^{\infty}\binom{12+s-1}{s}x^s$. So setting $s=r-12$ we see that the answer is

$$\binom{12+r-12-1}{r-12} = \binom{r-1}{r-12} = \binom{r-1}{11} \leftarrow \text{Answer}$$

(b) The appropriate generating function is

$$g(x) = (1+x^2+x^4+x^6+\dots)^{12}$$
$$= \sum_{s=0}^{\infty}\binom{12+s-1}{s}x^{2s} = \sum_{s=0}^{\infty}\binom{11+s}{s}x^{2s}$$

If r is odd the coeff of x^r is zero in the above expansion. If r is even then we can write $r=2s$ for some s, and the coeff of $x^r=x^{2s}$ is

$$\binom{11+s}{s} = \binom{11+s}{11} = \binom{11+r/2}{11}.$$

The answer therefore is

The # of ways to distribute r bags amongst 12 athletes so that each of them gets an even number of bags

$$= \underbrace{\begin{cases} 0 & \text{if } r \text{ is odd} \\ \binom{11+r/2}{11} & \text{if } r \text{ is even.} \end{cases}}_{\text{Answer}}$$

//

The following formulas are from Table 6.1, p.257 of the textbook.

Formulas (Table 6.1 of book)

(1) $\dfrac{1-x^{m+1}}{1-x} = 1+x+x^2+\dots+x^m$

(2) $1/(1-x) = 1+x+x^2+\dots$

(3) $(1+x)^n = 1+\binom{n}{1}x+\binom{n}{2}x^2+\dots+\binom{n}{r}x^r+\dots+\binom{n}{n}x^n$

(4) $(1-x^m)^n = 1-\binom{n}{1}x^m+\binom{n}{2}x^{2m}+\dots+(-1)^k\binom{n}{k}x^{km}$
$+\dots+(-1)^n\binom{n}{n}x^{2n}$

(5) $\dfrac{1}{(1-x)^n} = 1+\binom{1+n-1}{1}x+\binom{2+n-1}{2}x^2+\dots+\binom{r+n-1}{r}x^r+\dots$

(6) If $h(x)=f(x)g(x)$, where $f(x)=a_0+a_1x+a_2x^2+\dots$ and $g(x)=b_0+b_1x+b_2x^2+\dots$, then

$$h(x) = a_0b_0+(a_1b_0+a_0b_1)x+(a_2b_0+a_1b_1+a_0b_2)x^2+\dots$$
$$+(a_rb_0+a_{r-1}b_1+a_{r-2}b_2+\dots+a_0b_r)x^r$$
$$+\dots$$

Section 6.4 (Exponential Generating Functions)

These are more convenient when we want arrangements rather than selections.

Definition: An *exponential generating function* $g(x)$ for a_r, the number of arrangements with r objects, is a function with power series expansion

$$g(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \ldots + a_r \frac{x^r}{r!} + \ldots$$

How are these useful?

Example: Suppose we have to find the number of different r-letter words (arrangements) when the letters are chosen from an unlimited supply of a's, b's, and c's, and the word contains at least two a's.

To understand where we are going let us first look at the case $r = 4$. Before tackling this with exponential generating functions, let us do this in the old fashioned way. First we select and then we arrange. Possible selections are: $\{a,a,a,a\}$, $\{a,a,a,b\}$, $\{a,a,a,c\}$, $\{a,a,b,b\}$, $\{a,a,b,c\}$, and $\{a,a,c,c\}$ (remember the word contains at least two a's. Next we arrange these selections. The # of possible arrangements with these six selections is:

$$\frac{4!}{4!0!0!}, \frac{4!}{3!1!0!}, \frac{4!}{3!0!1!}, \frac{4!}{2!2!0!}, \frac{4!}{2!1!1!}, \frac{4!}{2!0!2!}$$

respectively. We have to sum this.

For general r, we claim that the exponential generating function (for the # of r-letter words formed from an unlimited # of a's, b's, c's and with at least two a's) is

$$g(x) = \left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)^2$$

exp. gen. fcn for a $1+x+x^2/2! + \dots$ = exp. gen fcn for b & c.

The coefficient of x^r in the above will be the number of formal products $(x^{e_1}/e_1!)(x^{e_2}/e_2!)(x^{e_3}/e_3!)$, where $e_1 + e_2 + e_3 = r$, $e_1 \geq 2$, $e_2, e_3 \geq 0$. Therefore the power series product is:

$$\left(\sum_{e_1+e_2+e_3=r} \frac{r!}{e_1!\,e_2!\,e_3!}\right)\frac{x^r}{r!} \qquad 2 \leq e_1, \quad 0 \leq e_2, e_3$$

The coefficient is exactly what we wanted for $\frac{r!}{e_1!\,e_2!\,e_3!}$ (where $e_1+e_2+e_3 = r \dots$) is the number of arrangements of a's, b's, c's with e_1 a's, e_2 b's, and e_3 c's.

Reminder: Exponential generating functions are good for arrangements and not for selections.

Example: Find the exponential generating function for a_r, the number of r arrangements without repetition of n objects.

Solution: Clearly $a_r = P(n,r) = \frac{n!}{(n-r)!}$ and so the answer is

$$g(x) = \sum_{r=0}^{n} \frac{n!}{(n-r)!}\,\frac{x^r}{r!}$$

But there is another way of doing this. Each object can occur only once. The generating function for one object is $1+x$. But we have n objects. So

$$g(x) = (1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r = \sum_{r=0}^{n} \frac{n!}{(n-r)!} \frac{x^r}{r!}$$

The two answers agree.

Distributing distinct objects

The point of this mini-section is to convince you that distributing distinct objects into different sets is an arrangement problem and not merely a selection problem. This is different from distributing identical objects. We have been through this argument before.

Suppose we have to distribute 20 distinct pens into five boxes. Say the pens are $P_1, P_2, \dots, P_{20}$ and the boxes are B_1, B_2, B_3, B_4, B_5. There are 5 possibilities for P_1, 5 for $P_2, \dots,$ and so on. The number of ways of distributing is 5^{20}.

There is another way of thinking about it. First write out the pens in a row:

$$P_1 \quad P_2 \quad P_3 \quad \dots \quad P_{18} \quad P_{19} \quad P_{20}$$

Suppose we have distributed the pens. Below each pen write the box name it has been assigned to. Say P_i gets assigned to X_i, where X_i is one of $B_1, B_2, B_3, B_4,$ or B_5. Then we have the following scheme:

$P_1 \quad P_2 \quad P_3 \quad \cdots \quad P_{18} \quad P_{19} \quad P_{20}$

$X_1 \quad X_2 \quad X_3 \quad \cdots \quad X_{18} \quad X_{19} \quad X_{20}$

We therefore have a sequence $X_1 X_2 X_3 \ldots X_{18} X_{19} X_{20}$ with each X_i being one of B_1, B_2, B_3, B_4, B_5 and with repetitions allowed. Clearly there are 5^{20} such sequences. If we require that each box has at least one pen, the answer is more complicated, but exponential functions come in useful as the following example shows.

Example: Find the exponential generating function for the number of ways to distribute r distinct apples amongst five people with at least one apple with each person.

Solution:

The answer is

$$g(x) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^5$$

Can be thought of as distributing $A_1, A_2, \ldots, A_r$ (distinct) into 5 boxes B_1, B_2, B_3, B_4, B_5 with at least one apple in each box. The number of such distributions is the same as the # of sequences $X_1 X_2 \ldots X_r$ where each X_i's are one of $B_1, \ldots, B_5$, and each B_i occurs at least once.

Start with x because we need at least one apple per person

$x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ = The exponential gen function for distributing r apples to one person with the person getting at least one apple.

The product with itself five times gives the required answer for five persons.

Section 6.5 (A summation method)

Suppose $A(x) = \sum a_n x^n$, $B(x) = \sum b_n x^n$, and $C(x) = \sum c_n x^n$ are three power series.

1. $B(x) = d A(x) \iff b_n = d a_n \ \forall n$ (d a constant)
2. $C(x) = A(x) + B(x) \iff c_n = a_n + b_n \ \forall n$.
3. $C(x) = A(x) B(x) \iff c_n = \sum_{i=0}^{n} a_i b_{n-i} \quad \forall n$.
4. $B(x) = x^k A(x) \iff b_i = 0$ for $i < k$ and $b_n = a_{n-k}$ for $i \geq k$.

Important trick!

Let $g(x) = \sum_{r=0}^{\infty} a_r x^r$ and $g^*(x) = \sum_{r=0}^{\infty} r a_r x^r$.

Then

$$g^*(x) = x\left(\frac{d}{dx} g(x)\right)$$

The above is fairly obvious. Leave the proof to you. See pp. 227-228 of the book if you cannot work it out.

Examples: Build generating function $h(x)$ for

1. $a_r = 2r^2$.

Soln: First, we know $\frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{r=0} x^r$

According to above:

$$x \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{r=0}^{\infty} r\, x^r$$

i.e

$$\frac{x}{(1-x)^2} = \sum_{r=0}^{\infty} r\, x^r$$

Applying the method one more time, we get

$$x \frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] = \sum_{r=0}^{\infty} r^2 x^r$$

i.e.

$$\sum_{r=0}^{\infty} r^2 x^r = x\left\{\frac{(1-x)^2 + 2x(1-x)}{(1-x)^4}\right\} = \frac{x(1+x)}{(1-x)^3}$$

Now multiply by 2 to obtain the generating function for $a_r = 2r^2$.

$$h(x) = \sum_{r=0}^{\infty} 2r^2 x^r = \frac{2x(1+x)}{(1-x)^3} \longleftarrow \text{generating function for } a_r = 2r^2.$$

2. $a_r = (r+1)r(r-1)$.

<u>Soln</u>: We could write $(r+1)r(r-1) = r^3 - r$ and obtain generating functions for r^3 and r and then subtract. That is fine. However it is easier to proceed as follows:

Let

$$f(x) = \frac{3!}{(1-x)^4} = 3!\,(1+x+x^2+\dots)^4 = \sum b_r x^r \quad \text{(say)}$$

We know that the coefficient of x^r in $(1+x+x^2+\dots)^4$ is $\binom{r+4-1}{r} = \binom{r+3}{r}$. Therefore the coefficient of x^r in $3!\,(1+x+x^2+\dots)^4$ is

$$b_r = 3!\binom{r+3}{r} = 3!\,\frac{(r+3)!}{r!\,3!} = \frac{(r+3)!}{r!} = (r+3)(r+2)(r+1).$$

Since a_r is given to be $(r+1)(r)(r-1)$ it is clear that

$b_r = a_{r+2}$ for $r \geq 0$. Thus:

$a_r = b_{r-2}$ for $r \geq 2$ (and zero for $r < 2$)

From Rule 4 at the beginning of this section, we get

$$h(x) = x^2 f(x).$$

It follows that

$$h(x) = \frac{3!\,x^2}{(1-x)^4}.$$

Generating function for $a_r = (r+1)\,r\,(r-1)$

More generally

$$\frac{(n-1)!}{(1-x)^n} = \sum_{r=0}^{\infty} a_r x^r$$

where

$$a_r = (n-1)! \binom{r+n-1}{r} = [r+(n-1)][r+(n-2)] \cdots (r+1)$$

Left to you to work out. Same technique as the one used for $a_r = 2r^2$.

Theorem: If $h(x)$ is a generating function where a_r is the coefficient of x^r, then $h^*(x) = h(x)/(1-x)$ is a generating function of the sums of the a_r's. That is,

$$h^*(x) = a_0 + (a_0+a_1)x + (a_0+a_1+a_2)x^2 + \cdots + \left(\sum_{i=0}^{r} a_i\right)x^r + \cdots.$$

Proof:

$$h^*(x) = \frac{h(x)}{1-x} = h(x)(1+x+x^2+\cdots)$$

$$= \left(\sum_{r=0}^{\infty} a_r x^r\right)\left(\sum_{r=0}^{\infty} x^r\right)$$

Now apply (3) in the list of rules at the beginning of this section, i.e., Section 6·5.

Example 1 (continued): Evaluate

$$2(1)^2 + 2(2^2) + 2(3^2) + \dots + 2(n^2)$$

Soln:

The generating function for $2r^2$ was found in Example 1 to be $h(x) = \dfrac{2x(1+x)}{(1-x)^3}$. According to the theorem, the generating function for $b_r = 2(0^2) + 2(1^2) + \dots + 2(r^2)$ is

$$h^*(x) = \frac{h(x)}{1-x} = \frac{2x(1+x)}{(1-x)^4}$$

$$= \frac{2x}{(1-x)^4} + \frac{2x^2}{(1-x)^4}$$

The coeff of x^n in $\dfrac{2x}{(1-x)^4}$ = The coeff of x^{n-1} in $\dfrac{2}{(1-x)^4}$

$$= 2\binom{(n-1)+4-1}{n-1} = 2\binom{n+2}{n-1}$$

The coeff of x^n in $\dfrac{2x^2}{(1-x)^4}$ = The coeff of x^{n-2} in $\dfrac{2}{(1-x)^4}$

$$= 2\binom{(n-2)+4-1}{n-2} = 2\binom{n+1}{n-2}$$

Thus the coeff of x^n in $h^*(x)$ is

$$2\binom{n+2}{n-1} + 2\binom{n+1}{n-2} = 2\binom{n+2}{3} + 2\binom{n+1}{3}$$

Thus

$$2(1^2) + 2(2^2) + 2(3^2) + \dots + 2(n^2) = 2\binom{n+2}{3} + 2\binom{n+1}{3}.$$

Example 2 (continued) : Evaluate

$$3\times 2\times 1 + 4\times 3\times 2 + \cdots + (n+1)n(n-1).$$

Solution :

Clearly we have been asked to find

$$b_n = \sum_{i=0}^{n} (i+1)(i)(i-1)$$

The generating function for b_r is, according to the theorem, equal to

$$h^*(x) = \frac{h(x)}{1-x}$$

where $h(x)$ is the generating function for $a_r = (r+1)r(r-1)$.
From Example 2, $h(x) = 6x^2/(1-x)^4$

$$h^*(x) = \frac{6x^2}{(1-x)^5}.$$

The coeff. of x^{n-2} in $6(1-x)^{-5}$ is what we have to find.
Thus

$$b_n = 6\binom{n-2+5-1}{n-2} = 6\binom{n+2}{4}$$

Answer

<u>Addendum</u>: The reason exponential generating functions work so well with arrangements has to do with the following observations.

Suppose a_r is the number of ways of arranging r objects from a collection A according to a procedure and b_r the number of arrangements according to some other procedure of r objects from B. Now suppose we wish to find all arrangements of n objects from A and B. First assume we regard objects from A as identical and those from B also as identical. Then the number of arrangements is with r objects from A and s objects from B is $\frac{(r+s)!}{r!\,s!}$. This is provided we think of all objects of A (respectively B) as identical to each other. If we drop the assumption, the correct count is $\frac{(r+s)!}{r!\,s!}a_r b_s$. If we want to arrange n objects from A and B, then the count is

$$c_n = \sum_{r+s=n} \frac{(r+s)!}{r!\,s!} a_r b_s = n! \sum_{r+s=n} \frac{1}{r!\,s!} a_r b_s . \qquad \text{———} (*)$$

Let $f(x) = \sum a_r \frac{x^r}{r!}$, $g(x) = \sum b_s \frac{x^s}{s!}$ and $h(x) = \sum c_n \frac{x^n}{n!}$ be the exponential generating functions for the three arrangement problems. We have that c_n satisfies $(*)$. On the other hand

$$\begin{aligned} f(x)g(x) &= \left(\sum a_r \frac{x^r}{r!}\right)\left(\sum b_s \frac{x^s}{s!}\right) \\ &= \sum_n \left(\sum_{r+s=n} \frac{a_r b_s}{r!\,s!}\right) x^n \\ &= \sum_n c_n \frac{x^n}{n!} \qquad \text{(via } (*)\text{)} \\ &= h(x). \end{aligned}$$

The formula $f(x)g(x) = h(x)$ is what helps us build generating functions.

For example in the problem of distributing r distinct apples amongst 5 persons so that each person gets one apple. Instead of five persons if we had to distribute the apples to one person then if $r \geq 1$

all the apples would be given to the person, but if $r=0$ no distribution is possible since we are forbidden from giving zero apples to anybody. So for one person the exponential generating function is $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$. It follows from the multiplicative property we derived above that the required generating function is

$$h(x) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)^5$$

Now

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

and hence

$$x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x - 1.$$

Therefore

$$\begin{aligned} h(x) &= (e^x - 1)^5 \\ &= e^{5x} - 5e^{4x} + 10e^{3x} - 10e^{2x} + 5e^x - 1 \\ &= \sum_{n\geq 0} \frac{5^n x^n}{n!} - 5\sum_{n\geq 0} \frac{4^n x^n}{n!} + 10\sum_{n\geq 0} \frac{3^n x^n}{n!} - 10\sum_{n\geq 0} \frac{2^n x^n}{n!} + 5\sum_{n\geq 0} \frac{x^n}{n!} - 1 \\ &= \sum_{n=0}^{\infty} \left[5^n - 5(4^n) + 10(3^n) - 10(2^n) + 5(1)\right] \frac{x^n}{n!} - 1 \end{aligned}$$

When $n=0$ $[5^n - 5(4^n) + 10(3^n) - 10(2^n) + 5(1)] = 1$

and hence

$$h(x) = \sum_{n=1}^{\infty} \left[5^n - 5(4^n) + 10(3^n) - 10(2^n) + 5(1)\right] \frac{x^n}{n!}$$

Thus the # of ways to distribute r distinct apples to 5 persons so that everyone gets at least one apple is

$$a_n = \begin{cases} 0 & \text{if } n=0 \\ 5^n - 5(4^n) + 10(3^n) - 10(2^n) + 5 & \text{for } n\geq 1. \end{cases}$$