March 15, 2018

Lecture 18

Section 6.1

Formal products: Suppose x and y are variables. Expressions of the form $yyxyxxx$, $xyxxyyyx$, $xxyx$ etc are called formal products of x and y and the formal product $yyxyxxx$ is considered to be different from the formal product $yxxxyyx$ (and similarly the formal product $xxyx$ is different from the formal product $xyxx$). However both $yyxyxxx$ and $yxxxyyx$ are said to represent x^4y^3.

In expanding the product of polynomials we first write out the expansion as a sum of formal products before "collapsing" into one all formal products which represent the same monomial. An example might help:

Example: The expansion of

$$(a+x)^n$$

can be regarded as the sum of formal products of the form $aaxxx\cdots a$, i.e., a string of length n consisting of a's and x's. For example

$$\begin{aligned}(a+x)^3 &= (a+x)(a+x)(a+x)\\ &= \left.\begin{aligned}&aaa + aax + axa + axx\\ &+ xaa + xax + xxa + xxx\end{aligned}\right\} \text{Sum of formal products.}\end{aligned}$$

Now collapse terms.

Coming back to the expansion of $(a+x)^n$, we have

$$(a+x)^n = \underbrace{(a+x)(a+x)(a+x)\cdots(a+x)}_{n\text{-times}}$$

The right side can be expanded as a sum of formal products, i.e., of strings of length n consisting of a's and x's. How many formal products represent $a^k x^{n-k}$? Clearly a formal product which represents $a^k x^{n-k}$ is a string with k a's and $n-k$ x's, and there are $\binom{n}{k}$ such strings. Thus we have

$$(a+x)^n = \sum_{k=0}^{n} \binom{n}{k} a^k x^{n-k}$$

i.e., we have the binomial theorem.

Notation: The notation

$$\begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix} \begin{Bmatrix} a_2 \\ b_2 \end{Bmatrix} \begin{Bmatrix} a_3 \\ b_3 \end{Bmatrix}$$

will denote a formal product of length 3 which has either a_1 or b_1 as its first term; a_2 or b_2 as its second term; a_3 or b_3 as its third term. Thus the symbol could mean any of the eight formal products:

$a_1a_2a_3$, $a_1a_2b_3$, $a_1b_2a_3$, $a_1b_2b_3$,

$b_1a_2a_3$, $b_1a_2b_3$, $b_1b_2a_3$, $b_1b_2b_3$.

Similarly one can assign a meaning to symbols of the form

$$\begin{Bmatrix} a_1 \\ b_1 \\ c_1 \end{Bmatrix} \begin{Bmatrix} a_2 \\ b_2 \\ c_2 \end{Bmatrix} \cdots \begin{Bmatrix} a_n \\ b_n \\ c_n \end{Bmatrix}$$

etc.

Generating functions: Suppose a_r is the number of ways to select r objects in a procedure. Then the generating function $g(x)$ for a_r is

$$g(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_r x^r + \dots + a_n x^n.$$

If the function has infinite terms, it is called a power series.

Thus $g(x) = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ is the generating function for $a_r = \binom{n}{r}$.

Example: Let us expand

$$g(x) = (1+x+x^2)^4.$$

On expanding we have deal with formal products of the type

$$\begin{Bmatrix} 1 \\ x \\ x^2 \end{Bmatrix} \begin{Bmatrix} 1 \\ x \\ x^2 \end{Bmatrix} \begin{Bmatrix} 1 \\ x \\ x^2 \end{Bmatrix} \begin{Bmatrix} 1 \\ x \\ x^2 \end{Bmatrix}$$

A typical term is of the form

$$x^{e_1} x^{e_2} x^{e_3} x^{e_4} \qquad 0 \le e_i \le 2.$$

Suppose one wishes to find the coefficient of x^5 in the expansion of $(1+x+x^2)^4$. The problem becomes the same as finding the number of non-negative integer solutions of

$$e_1 + e_2 + e_3 + e_4 = 5 \qquad 0 \le e_i \le 2.$$

This is the same the problem of distributing five identical objects into four boxes with no more than two objects in each box.

Similarly, the coefficient a_r of x^r in $(1+x+x^2)^4$ is the number of formal products $x^{e_1}x^{e_2}x^{e_3}x^{e_4}$, $0 \le e_i \le 2$, such that the sum of the e_i's is r, i.e.,

$$e_1+e_2+e_3+e_4 = r \qquad 0 \le e_i \le 2.$$

The problem then is to put r identical objects into four boxes, making sure that no box has more than two objects.

$$\boxed{\text{Read examples 1, 2, 3, 4 from Section 6-1. They are easy.}}$$

Example : Build a generating function for a_r, the number of integer solutions to :

$$e_1+e_2+e_3+e_4 = r, \qquad e_i \ge 0.$$

Solution :

Let $g(x) = (1+x+x^2+\dots)(1+x+x^2+\dots)(1+x+x^2+\dots)(1+x+x^2+\dots)$

The coefficient of x^r is the number of formal products of the form

$$\begin{Bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^4 \\ \vdots \end{Bmatrix} \cdot \begin{Bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^4 \\ \vdots \end{Bmatrix} \cdot \begin{Bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^4 \\ \vdots \end{Bmatrix} \cdot \begin{Bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^4 \\ \vdots \end{Bmatrix} \quad \text{or } x^{e_1}x^{e_2}x^{e_3}x^{e_4}, \ e_i \ge 0$$

such that the sum of the exponents e_1, e_2, e_3, e_4 is r.

In other words the coefficient of x^r is a_r.

Now $(1+x+x^2+\dots) = \frac{1}{1-x}$. So we can write

$g(x) = \frac{1}{(1-x)^4}$. Either answer, $g(x) = (1+x+x^2+\dots)^4$ or

$g(x) = \frac{1}{(1-x)^4}$, is correct.

Example: Same as the above problem but with $0 \le e_i \le 7$.

Solution: $g(x) = (1+x+x^2+x^3+x^4+x^5+x^6+x^7)^4$.

Example: Build a generating function for a_r, the number of distributions of r identical objects into five different boxes with between four and eight objects in each box.

Solution:

Check that

$$g(x) = (x^4+x^5+x^6+x^7+x^8)^5$$

works. You supply the reasons.

Section 6.3 (Partitions)

Definition: A *partition* of a positive integer r is a collection of positive integers whose sum is r.

Partitions of 5:

$1+1+1+1+1$, $1+1+1+2$, $1+1+3$, $1+2+2$, $1+4$, $2+3$, 5

"trivial" partition.

Examples : Build generating functions for a_r

1. a_r = # of partitions of r.

Soln: A partition of r is specified by the number of 1's, the number of 2's, etc occurring in the partition. If e_k is the number of k's in the partition, then

$$1\cdot e_1 + 2\cdot e_2 + 3\cdot e_3 + \dots + k e_k + \dots + r e_r = r.$$

A little thought shows that

$$\begin{aligned} g(x) = &\ (1+x+x^2+\dots+x^n+\dots) \\ &\cdot (1+x^2+x^4+\dots+x^{2n}+\dots) \\ &\cdot (1+x^3+x^6+\dots+x^{3n}+\dots) \\ &\ \vdots \\ &\cdot (1+x^k+x^{2k}+\dots+x^{kn}+\dots) \\ &\ \vdots \end{aligned}$$

$$= \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdot \dots \cdot \frac{1}{1-x^k} \dots \quad \text{(infinite product).}$$

Remark : For partitions up to $r = m$, only need the first m factors.

2. a_r = # of ways of expressing r as a sum of <u>distinct</u> integers.

(For example, for 5 the partitions by distinct integers are 1+4, 2+3, and 5.).

Soln : $g(x) = (1+x)(1+x^2)(1+x^3)(1+x^4)\dots(1+x^k)\dots$

4. a_r = # of ways to choose 2¢, 3¢ and 5¢ stamps adding to a net value of r¢.

<u>Solution</u> : We want e_1, e_2, e_3 such that

$$2e_1 + 3e_2 + 5e_3 = r, \quad e_i \geq 0.$$

The appropriate gen. function is

$$g(x) = (1+x^2+x^4+x^6+\cdots)(1+x^3+x^6+x^9+\cdots)\cdot$$
$$\cdot\ (1+x^5+x^{10}+x^{15}+\cdots)$$

<u>Ferrer's diagram</u>: The <u>Ferrer's diagram</u> of a partition of positive integer r is a display of r dots in a set of rows listed in order of decreasing size.

The Ferrer's diagram for the partitions 1+2+2+3+7 of 15 is:

7 dots   • • • • • • •

3 dots   • • •

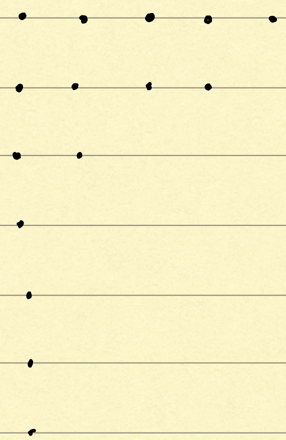
2 dots   • •

2 dots   • •

1 dot   •

Suppose you transpose the rows and the columns of a Ferrer's diagram for a partition of r. Then you get a new Ferrer's diagram for a (possibly) different partition of r, and this new diagram is called the <u>conjugate</u> of the original Ferrer's diagram.

The conjugate of the Ferrer's diagram for 15 given above is:



Conjugate Ferrer's diagram to the one above. The corresponding partition of 15 is $1+1+1+1+2+4+5$

<u>Remark</u> : If D_1 and D_2 are two Ferrer's diagrams such that their conjugates are equal then $D_1 = D_2$.

Using this it is easy to see that the number of partitions of a positive integer r into a sum of m positive integers is the same as the number of partitions of r as a sum of positive integers, the largest of which is m (use conjugate Ferrer's diagrams).

Start reading Section 6.2