Mar 13, 2018 Lecture 17

Section 5.5 (Binomial Identities)

Theorem (The Binomial Theorem):

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{k}x^k + \dots + \binom{n}{n}x^n$$

Proof: Consider the expansion of $(a+x)^n$

$$(1+x)^n = \underbrace{(1+x)(1+x)\cdots(1+x)}_{n\text{-times}} = b_0 + b_1x + \dots + b_kx^k + \dots + b_nx^n.$$

Let k be an integer between 0 and n ($0 \le k \le n$). We wish to understand the coefficient b_k of x^k in the expansion. The x^k appears by the multiplication of k x's from the n-factors above. There are $\binom{n}{k}$ ways of picking k factors from the n above, and this means $b_k = \binom{n}{k}$. q.e.d.

There are other proofs using induction.

Setting $x=1$ in the formula we recover the formula

$$\boxed{\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n}$$

Note that there is a symmetry and we have

$$\binom{n}{k} = \binom{n}{n-k}$$

as we observed earlier.

Example: Show $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Solution:

Partition your n objects $a_1, a_2, \ldots, a_n$ into two sets; the set $S = \{a_1, \ldots, a_{n-1}\}$ and the set $\{a_n\}$.

$$\overbrace{a_1 \quad a_2 \quad \cdots \quad a_{n-1}}^{S} \;\Big|\; a_n$$

There are two mutually exclusive ways of picking k objects from $a_1, \ldots, a_n$. Either one can pick k objects from S (there are $\binom{n-1}{k}$ ways of doing this) or pick $k-1$ objects from S and a_n to the $k-1$ objects to get a collection of k objects (there are $\binom{n-1}{k-1}$ ways of doing this.

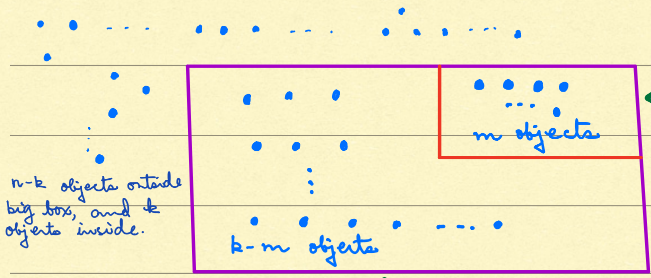
This gives the result. q.e.d.

Example: Show that $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$

Solution:

The left side counts the number of ways to pick k objects (from n objects) to put in a box, and then from the objects in the box, picking m objects to be put in a corner of the box. Equally well, one could first pick m

objects (from n) to put in the corner of the box and then pick the remaining $k-m$ objects to put in the box from the remaining $n-m$ objects. This process is counted by the right side. (The picture below may help.)



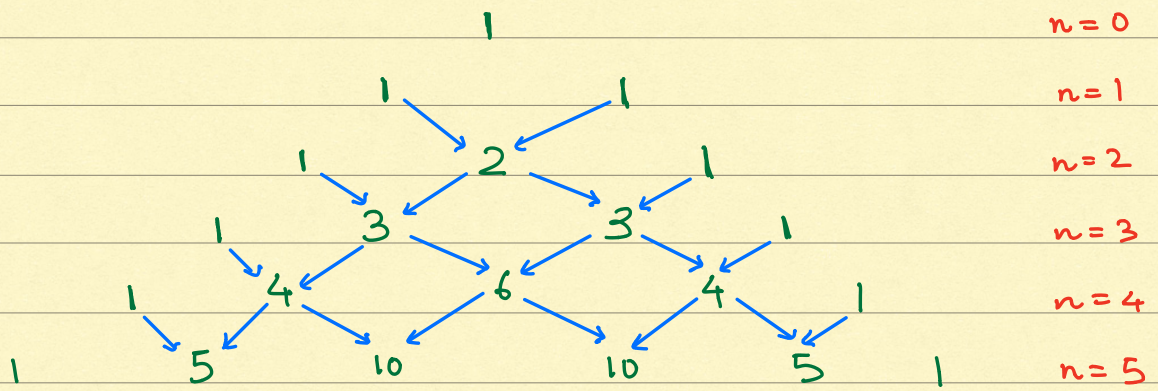
$\binom{n}{m}\binom{n-m}{k-m}$ m objects chosen from n and put in a corner of a big box, and then another k-m objects chosen from the remaining n-m objects and put in other parts of the box.

Equivalent ways of sorting

$\binom{n}{k}\binom{k}{m}$ k objects chosen from n and then m objects from these separated out and put in a corner.

The above method can also be regarded as the committee selection method. Suppose we had to choose a committee of k members from n people and once the committee is formed select a subcommittee of m members from the committee. There are $\binom{n}{k}\binom{k}{m}$ ways of doing this. However, we could also do the same sort of selection in a different way. We could select the m members of the subcomittee first and then the remaining $k-m$ members of the committee from the remaining pool of $n-m$ members. When we count the number of ways we can do this, we arrive at $\binom{n}{m}\binom{n-m}{k-m}$. It follows that $\binom{n}{k}\binom{k}{m} = \binom{n-m}{k-m}$.

There is yet another way of thinking about all this via due to Polya — the so called Block-walking method. To motivate that we recall **Pascal's triangle** for binomial coefficients. Recall that the n-th row of the triangle lists the binomial coefficients $\binom{n}{k}$ as k varies from 0 to n. The two ends of the n^{th} row are $\binom{n}{0}$ and $\binom{n}{n}$ and both these numbers are 1. The n^{th} row has $n+1$ entries, since k varies from 0 to n (not from 1 to n). The Δ is



The blue arrow indicate the following. An entry which has two arrows pointing to it is the sum of the two entries which are at the tails of the two arrows. This gives an easy way to build binomial coefficients at least for small n.

The question is — why does it work? It is clearly based on the identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ which we have proven earlier. We will prove the identity in a different way making Pascal's Δ very transparent.

<u>Block-walking model</u> : Consider the following picture (Figure 5.1 on p. 229 of the textbook.)

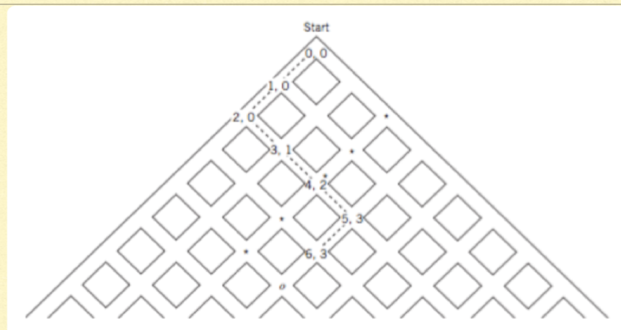


Figure 5.1 on page 229 of the textbook

You are supposed to think of the above as a map of streets. Imagine a person walking along the streets. At each intersection the person turns either right or left (for simplicity, "right" is YOUR right, not that of the walker, and similarly "left" is YOUR left).

Suppose the walker starts at the corner represented by the top of the triangle, and after making a series of turns reaches a corner C. Suppose the walker traversed n blocks and made right turns along the way. Check that any other route the walker takes requires traversing n blocks and making k right turns. We label the corner C as (n, k).

In the picture $(0,0)$ (the starting point), $(1,0)$, $(2,0)$, $(3,1)$, $(4,2)$, $(5,3)$, and $(6,3)$ are illustrated.

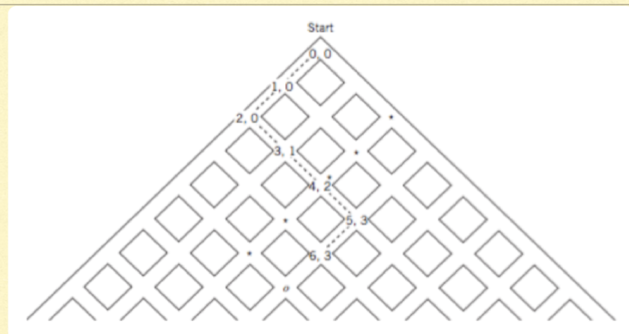


Figure 5.1 on page 229 of the textbook

The walker's route going from $(0,0)$ to a corner C after walking through n blocks can be coded as a sequence of R's (for right turns) and L's (for left turns) of length n. For example RLLRLRR denotes a route from $(0,0)$ starting with a right turn at that corner, a left turn at the next corner, another left turn at the one after, followed by a right turn, followed by a left turn followed by two right turns. This took 7 "steps" and 4 right turns and the corner one ends up at is $(7,4)$. The route is $(0,0) \longrightarrow (1,1) \longrightarrow (2,1) \longrightarrow (3,1) \longrightarrow (4,2) \longrightarrow (5,2) \longrightarrow (6,3) \longrightarrow (7,4)$ (check this to test your understanding).

How many ways are there of getting from $(0,0)$ to (n,k)? Since each way is described by a sequence of L's and R's of length n with k R's, it follows that there are $\binom{n}{k}$ ways of getting from $(0,0)$ to (n,k). Now the last but one corner the walker passes before reaching (n,k) is either $(n-1,k)$ or $(n-1,k-1)$ (the walker makes a left turn in the former case and a right turn in latter case). It follows that

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}.$$

The nice thing about the block-walking model is that the street map is shaped exactly like Pascal's Δ, and the street corners are precisely where entries are made in Pascal's Δ. Moreover, the way we have chosen to label corners, i.e., by (n,k) where n is the number of walks blocked and k the number of right turns, we see that from the vertex $(0,0)$ to (n,k) there are $\binom{n}{k}$ paths. This allows us to prove many binomial identities.

Example : Show that $\binom{r}{r} + \binom{r+1}{r} + \dots + \binom{n-1}{r} + \binom{n}{r} = \binom{n+1}{r+1}$

Solution : There are two ways of seeing this. The first is as follows. Let $R = \{1, 2, \dots, n, n+1\}$. The number of subsets S of R with exactly $r+1$ elements is the right hand side of the identity we have been asked to prove, i.e., $\binom{n+1}{r+1}$. We can break this count into many cases. We could look for all subsets S with $r+1$ elements such that the largest element in S is k. Note that k cannot be less than $r+1$ since S has $r+1$ elements. So the cases are $k=r+1$, $k=r+2$, $\ldots$, $k=n$, $k=n+1$.

Now the number of subsets S of R with $r+1$ elements which have k as the largest element, is clearly the same as the number of subsets of $\{1,2,\ldots,k-1\}$ which has r elements. This is $\binom{k-1}{r}$. Thus the count is $\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \binom{r}{r} + \binom{r+1}{r} + \ldots + \binom{n}{r}$. This proves the assertion.

The other way of doing this is the block-walking method. I will give the briefest indication of how this is done and leave it to you to flesh out the details (or look at Example 2 on p. 231 of the book). Clearly the RHS of our identity the number of paths from $(0,0)$ to $\binom{n+1}{r+1}$. After the walker makes the $(r+1)$st right turn he/she is at the corner $(k, r+1)$. The corner the walker was at in the $(k-1)$st step is clearly $(k-1, r)$. Once the walker reaches $(k, r+1)$ there is only path to $(n+1, r+1)$ namely left turns all the rest of the way. The number of ways the walker reaches $(k-1, r)$ is $\binom{k-1}{r}$. The rest of the argument is left to you.

Example: Show $\sum_{k=0}^{m} \binom{m}{k}\binom{n}{r+k} = \binom{m+n}{m+r}$

Solution: Suppose we have $m+n$ objects, say $\{1, 2, \ldots, m+n\}$ and we have pick $m+r$ elements from it. Suppose that in the process we have picked $m-k$ elements from $\{1, \ldots, m\}$ and $r+k$ objects from $\{m+1, \ldots, m+n\}$. There are $\binom{m}{m-k}\binom{n}{r+k}$ ways of doing this, and k ranges from 0 to m. Thus

$$\binom{m+n}{m+r} = \sum_{k=0}^{m} \binom{m}{m-k}\binom{n}{r+k}.$$

Now we know that $\binom{m}{m-k} = \binom{m}{k}$. This gives the result.