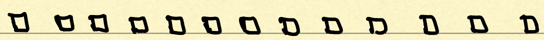
Mar 8, 2018

Lecture 16

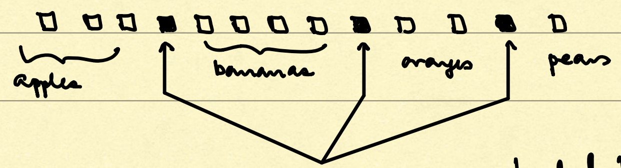
Example: Suppose we have to pick 10 fruits from a store containing apples, bananas, oranges, and pears. Assume that the apples are identical, as are the bananas, oranges, and pears and that the store has a large stock of each. How many ways are there of selecting 10 fruits?

Solution

Consider the following 13 squares



labelled 1, 2, 3, ..., 13. Now pick any 3 of them and shade them black. For example, as below, squares 4, 9, and 12



Selected squares shaded black.

This divides the unshaded squares into four regions. This gives us a way of selecting the number of apples, bananas, oranges and pears; in this case 3 apples, 4 bananas, 2 oranges, and one pear. It is clear that every selection of three squares out of 13, results in such 4 regions, and an assignment of apples, bananas, oranges, and pears making for a total of 10 fruits.

Conversely, suppose we have selected 10 fruits of which a_1 are apples, a_2 are bananas, a_3 are oranges, a_4 are pears so that $a_1 + a_2 + a_3 + a_4 = 10$, then the three squares out of the 13 that give this assignment are the squares $a_1 + 1$, $a_2 + 2$, and $a_3 + 3$.

So there is a one-to-one correspondence between a selections 10 fruits from apples, bananas, oranges, and pears, and a selection of 3 squares from 13.

Thus the answer is $\binom{13}{3}$.

The technique gives the following generalisation.

Theorem 2 (of Section 5.3) : The number of selections with repetitions of r objects chosen from n types of objects is $\binom{r+n-1}{r}$

Proof:

Pick $n-1$ elements from the set $\{1, 2, 3, \dots, r+n-1\}$, say $b_1, b_2, b_3, \dots, b_{n-1}$.

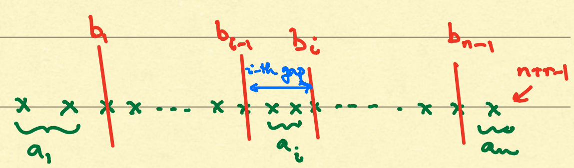
Let

$$\left.\begin{aligned}
a_1 &= b_1 - 1 \\
a_2 &= b_2 - b_1 - 1 \\
a_3 &= b_3 - b_2 - 1 \\
&\vdots \\
a_{n-1} &= b_{n-1} - b_{n-2} - 1 \\
a_n &= r + n - 1 - b_{n-1}
\end{aligned}\right\} \quad (*)$$

The idea is in the pic below. Draw $r+n-1$ crosses and pick b_{n-1} of them. This leaves n gaps and there are r spots in these gaps. The length of the i-th gap is a_i.

Then

$$a_1 + a_2 + \dots + a_n = r.$$



we pick a_i number of objects of type i.

Conversely suppose we have chosen r objects from n-types of objects; say a_1 objects of type 1, a_2 of type 2, ..., a_n of type n. The system of equations (*) gives us $n-1$ numbers $b_1, \ldots, b_{n-1}$ from the set $\{1, 2, \ldots, n+r-1\}$. In fact

$$b_1 = a_1 + 1$$

$$b_2 = a_1 + a_2 + 2$$

$$\vdots$$

$$b_{n-1} = a_1 + a_2 + \cdots + a_{n-1} + (n-1).$$

Since $a_1 + \cdots + a_{n-1} \leq a_1 + \cdots + a_{n-1} + a_n = r$, we have $a_1 + \cdots + a_{n-1} \leq r$, i.e., $b_{n-1} \leq r + n - 1$.

Thus there is a one-to-one correspondence between selection of r objects from n type of objects (with repetition allowed), and selections of $n-1$ elements from the set $\{1, 2, \ldots, r+n-1\}$.

Thus:

The # of selections with repetitions of r objects chosen from n types of objects

$$= \binom{r+n-1}{n-1}$$

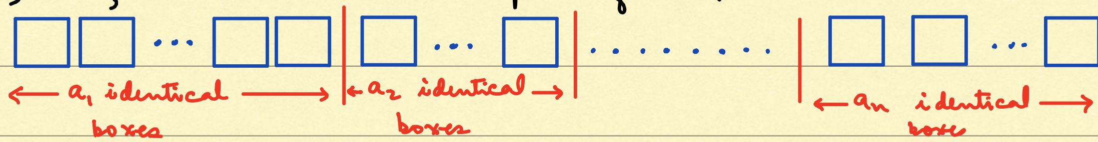
$$= \binom{r+n-1}{r}$$

q.e.d.

Variant of above result: Suppose we have to distribute r identical objects into n boxes. How many ways are there of doing this?

Let the boxes be $B_1, B_2, \ldots, B_n$. Suppose we have put

a_1 objects into B_1, a_2 into B_2, ..., a_n into B_n. We can regard this as selecting a_1 identical copies of B_1, a_2 identical copies of B_2, . . ., and a_n identical copies of B_n.



Since $a_1 + \dots + a_n = r$, this is really the old problem we solved. As a result we conclude the following.

Identical Objects (p.215, § 5.4) The process of distributing r identical objects into n different boxes is equivalent to choosing an (unordered) r box names with repetition from among n choices of boxes. Thus there are

$$\binom{r+n-1}{r} = \frac{(r+n-1)!}{r!\,(n-1)!}$$

distributions of the r identical objects into n boxes.

Example: How many ways are there of distributing 10 identical oranges amongst 6 boys?

Solution: According to our analysis the answer is

$$\binom{10+6-1}{10} = \binom{15}{10} = \binom{15}{5} = \frac{15 \times 14 \times 13 \times 12 \times 11}{5!} \text{ ways}$$

Example: Do the above example, but this time ensure that every boy gets at least one orange.

Solution: First give each boy one orange. Now we have four oranges to be distributed amongst six boys. The number of ways this can be done is $\binom{4+6-1}{4} = \binom{9}{4} = \frac{9\times 8\times 7\times 6}{4!}$ ways.

Example (Example 5 of Section 5.4): Show that the number of ways to distribute r identical balls into n distinct boxes with at least one ball in each box is $\binom{r-1}{n-1}$. With at least r_1 balls in the first box, r_2 balls in the second box, ..., and at least r_n balls in the n^{th} box, the number is $\binom{r-r_1-\dots-r_n+n-1}{n-1}$.

Solution:

The first part can be done exactly as in the previous example. Put one ball in each box. Now we have $r-n$ balls to be distributed in n boxes. This can be done in $\binom{(r-n)+n-1}{r-n}$ ways.

Now
$$\binom{(r-n)+n-1}{r-n} = \binom{(r-n)+n-1}{n-1} = \binom{r-1}{n-1}$$

as asserted.

One can extend this reasoning to the second part of the problem. Put r_1 balls in the first box, r_2 balls in the second box, ..., r_n balls in the n^{th} box. Now we have $r-r_1-r_2-\dots-r_n$ balls to be distributed into n boxes. This can be done in

$$\binom{(r-r_1-\dots-r_n)+n-1}{r-r_1-\dots-r_n} = \binom{(r-r_1-\dots-r_n)+n-1}{n-1} \text{ ways.}$$

What the above really shows is the following:-

Let r and n be positive integers. The number of nonnegative integer solutions to

$$x_1 + \ldots + x_n = r$$

is $\binom{n+r-1}{r}$.

One way of seeing this is to note that each non-neg integer soln of the equation gives a way of distributing r identical objects into n boxes and vice-versa. In fact if $(a_1, \ldots, a_n)$ is a solution of $x_1 + \ldots + x_n = r$, then we can put a_1 objects into the first box, a_2 objects into the second, and so on. Conversely, if we have a way of distributing r identical objects into n boxes, and in this distribution we have a_i objects in the i-th box, then $a_1, \ldots, a_n$ are non-negative integers and $a_1 + \ldots + a_n = r$, i.e., $(a_1, \ldots, a_n)$ is a non-neg integer solution of $x_1 + \ldots + x_n = r$.

The reasoning can be extended to give:

Suppose n, r are positive integers, $r_1, r_2, \ldots, r_n$ nonnegative integers such that $r_1 + \ldots + r_n \le r$, then the number of non-negative integer solutions of $x_1 + \ldots + x_n = r$ with $x_i \geq r_i$ for $1 \le i \le n$ is

$$\binom{(r - r_1 - \ldots - r_n) + n - 1}{n-1}.$$

The formula works because the problem can be re-cast as the problem of finding the number of ways r identical objects can be distributed in n boxes so that there are at least r_i objects in the i-th box, $1 \le i \le n$.

Example 6 from Section 5.4: How many integer solns are there to the eqn $x_1 + x_2 + x_3 + x_4 = 12$, with $x_i \geq 0$? How many solns with $x_i \geq 1$? How many solns with $x_1 \geq 2$, $x_2 \geq 2$, $x_3 \geq 4$, $x_4 \geq 0$?

Solution: In this example, $n=4$, $r=12$. For the first question the answer is $\binom{n+r-1}{r} = \binom{15}{12} = \binom{15}{3} = \frac{15 \times 14 \times 13}{3 \times 2} = 5 \times 7 \times 13 = 455$

Answer

For the second we have $r_1 = r_2 = \dots = r_n = 1$, and hence the number of solutions with this constraint is

$$\binom{(r-r_1-\dots-r_n)+n-1}{n-1} = \binom{(12-4)+3}{3} = \binom{11}{3}$$

$$= \frac{11 \times 10 \times 9}{3!} = 11 \times 5 \times 3 = 165.$$

Answer

For the third question, from our formula we have that the # of solns with the given constraint is

$$\binom{(12-2-2-4-0)+3}{3} = \binom{7}{3} = \frac{7 \times 6 \times 5}{3!} = 35. \leftarrow \text{Answer}$$

Example: How many non-negative integer solutions are there to the equation $a+b+c+d+5e = 15$?

Solution: Note that if in any solution $e \geq 4$, then $5e \geq 20 > 15$, and one or more of $a, b, c,$ or d will have to be negative, something the question does not allow. Hence $e = 0, 1, 2$, or 3.

Case 1: $e=0$. In this case we have to find # of solns of $a+b+c+d=15$. The answer is $\binom{18}{3}$.

Case 2: $e=1$. In this $a+b+c+d=15-5e=10$. We have to find the # of solns of $a+b+c+d=10$. The answer is $\binom{13}{3}$.

Case 3: $e=2$. Now $5e=10$, and we have to find the # of solns of $a+b+c+d=5$. The answer is $\binom{8}{3}$.

Case 4: $e=3$. Since $5e=15$, we have to find the # of solns of $a+b+c+d=0$, and the answer is $\binom{3}{3}=1$. ← It is obvious that there is only one soln of $a+b+c+d=0$ with a,b,c,d non-negative integers.

Adding, we get:

$$\text{\# of non-negative integer solutions} = \boxed{\binom{18}{3}+\binom{13}{3}+\binom{8}{3}+\binom{3}{3}}.$$

↑ Answer

See also Example 7 in Section 5.4 ("Ingredients for a Witch's Brew")

Example 8 of Section 5.4: What fraction of binary sequences of length 10 consists of a positive number of 1's, followed by a positive number of 0's, followed by a positive number of 1's, followed by a positive number of 0's?

Solution:

Here are two examples of such sequences:

1100111000, 1000011110

Here is something which is NOT AN EXAMPLE of the kind of sequences we are looking at: 0010001111.

The total of all binary 10 digit sequences is $2^{10}=1024$.

For any binary sequence of the kind we are looking at, we have four segments, the first of 1's, the next of 0's, the next of 1's, and a final segment of 0's. Let the lengths of these sequences be a_1, a_2, a_3, and a_4 respectively. Then since the lengths of these segments are positive, $a_1 \geq 1$, $a_2 \geq 1$, $a_3 \geq 1$, $a_4 \geq 1$. Also $a_1+a_2+a_3+a_4=10$. Conversely, given integers a_1, a_2, a_3, a_4 such that $a_i \geq 1$, $1 \leq i \leq 4$ and $a_1+a_2+a_3+a_4=10$, we can form a 10 digit binary sequence of the required form, with segment lengths being a_1, a_2, a_3, and a_4. So the answer is $\binom{(10-4)+3}{3} = \binom{9}{3} = 84$.

The required fraction is:

$$\frac{84}{1024}.$$

This is approximately 8% of all 10 digit binaries.

<u>Example</u> (<u>Like Example 9 in § 5.4</u>): How many ways are there of arranging K, L, M, N, Z, Z, Z, Z, Z, Z, Z (seven Z's) if we cannot have two consecutive letters from the set $\{K, L, M, N\}$?

<u>Solution</u>: Let us introduce a new symbol R to stand for any element in $\{K, L, M, N\}$. Consider the number of ways of ~~having~~ distributing the seven Z's into five segments as in the following picture

with $a_1 \geq 0,\ a_2 \geq 1,\ a_3 \geq 1,\ a_4 \geq 1,\ a_5 \geq 0,\ a_1+a_2+a_3+a_4+a_5 = 7$

$$\underset{\downarrow}{\overset{a_1\ Z\text{'s}}{\underline{\qquad}}}\ R\ \underset{\downarrow}{\overset{a_2\ Z\text{'s}}{\underline{\qquad}}}\ R\ \underset{\downarrow}{\overset{a_3\ Z\text{'s}}{\underline{\qquad}}}\ R\ \underset{\downarrow}{\overset{a_4\ Z\text{'s}}{\underline{\qquad}}}\ R\ \underset{\downarrow}{\overset{a_5\ Z\text{'s}}{\underline{\qquad}}}$$

The R's in the middle stand for the letters from $\{K, L, M, N\}$. If we have an arrangement as above, we can fill in the places taken up by the four R's by K, L, M, N in $4! = 24$ ways. For example suppose we have $a_1 = 1,\ a_2 = 1,\ a_3 = 2,\ a_4 = 3,\ a_5 = 0$.

$$\underline{Z}\ R\ \underline{Z}\ R\ \underline{ZZ}\ R\ \underline{ZZZ}\ R\ \underline{\quad} \leftarrow \text{no } Z\text{'s.}$$

Pick an arrangement for $\{K, L, M, N\}$, say MLKN. Then fill the R's according to this arrangement and we get Z M Z L ZZ K ZZZ N. The fact that $a_2 \geq 1, a_3 \geq 1, a_4 \geq 1$ ensure that we cannot have consecutive letters from $\{K, L, M, N\}$. The number of solns of $a_1+a_2+a_3+a_4+a_5 = 7$ with $a_1 \geq 0,\ a_2 \geq 1,\ a_3 \geq 1,\ a_4 \geq 1,\ a_5 \geq 0$ is

$$\binom{(7-0-1-1-1-0)+5-1}{5-1} = \binom{8}{4} = 70.$$

However for every such arrangement of Z's and R's we have $4!$ arrangements with Z's and K, L, M, N. Thus the number of required arrangements is

$$4!\binom{8}{4} = (4!)(70) = (24)(70) = 1680. \leftarrow \text{Answer}$$

IMPORTANT* : READ Example 4 from Section 5.4.
(You were asked to earlier too)

What we have seen is that the following three problems are equivalent.

Three Equivalent Formulations

1. The number of ways of selections of r objects from n types of objects.
2. The number of ways of distributing r identical objects into n distinct objects.
3. The number of non-negative integer solutions of the equation $e_1 + e_2 + \dots + e_n = r$

The answer in every case is

$$\binom{n+r-1}{r}$$

If $r_1, \dots, r_n \geq 0$ with $r_1 + \dots + r_n \leq r$, then the problem of selecting r objects from n types with at least r_i objects of the i^{th} type is equivalent to the problem of distributing r identical objects into n distinct boxes with at least r_i objects in the i^{th} box, and both are equivalent to the problem of finding non-negative integer solutions of $e_1 + \dots + e_n = r$ with the constraint $e_i \geq r_i$ for all i. In all cases the number of ways of achieving what is required is:

$$\binom{r - r_1 - r_2 \dots - r_n + n - 1}{n-1}$$