Mar 6, 2018

Lecture 15

3. (Binary sequences) How many 12 digit binary sequences are there with seven 1's and five 0's?

Solution

Out of the 12 spots for digits □ □ □ □ □ □ □ □ □ □ □ □, we can pick any 5 and put 0's there and 1's elsewhere. Thus the answer is $\binom{12}{5}$. Equally well, we could pick

7 spots out of the 12, and put 1's there. By this count the answer is $\binom{12}{7}$. However, as is obvious $\binom{12}{5} = \binom{12}{7}$ and both equal $\frac{12!}{7!\,6!}$.

Answer

Note that in general

$$\binom{n}{r} = \binom{n}{n-r}$$

and both sides equal $n!/r!\,(n-r)!$. A combinatorial way of seeing this is: Picking r objects out of n is also a way of picking n-r objects out of n (the "remaining objects") and vice-versa.

Aside. Subsets of a finite set: Given a finite set E of n elements, say $E = \{e_1, e_2, \ldots, e_n\}$, one can form an n-digit binary sequence $x_S = x_1 x_2 \ldots x_n$ for every subset S of E as follows: If $e_i \in S$, set $x_i = 1$, and $e_i \notin S$, set $x_i = 0$. Conversely given an n-digit binary sequence $x_1 x_2 \ldots x_n$ we can form a subset S of E by defining S to be the collection of e_i such that $x_i = 1$.

$$S = \{ e_i \in E \mid x_i = 1 \}$$

Since there are 2^n n-digit sequences we have:

$$\boxed{\#\text{ of subsets of a set of } n \text{ elements} = 2^n}$$

For example, the empty set corresponds to the binary sequence $00\ldots0$, and the whole set E corresponds to $11\ldots1$.

Now we know that the number of subsets of E consisting of r elements is $\binom{n}{r}$ by definition of $\binom{n}{r}$. Thus

$$\begin{aligned}
\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} &= \#\text{ of subsets of } E \text{ with no elements} \\
&\quad + \#\text{ of subsets of } E \text{ with 1 element} \\
&\quad + \#\text{ of subsets of } E \text{ with 2 elements} \\
&\quad + \\
&\quad \vdots \\
&\quad + \#\text{ of subsets of } E \text{ with } n \text{ elements} \\
&= \#\text{ of subsets of } E = 2^n \text{ (from display above)}
\end{aligned}$$

We thus have the formula

$$\boxed{\sum_{r=0}^{n} \binom{n}{r} = 2^n}$$

Can also be done using the binomial theorem (later).

4. (Poker Probabilities)

(a) How many 5-card hands (subsets) can be formed from a standard 52-card deck?

(b) If a 5-card hand is chosen at random, what is the probability of obtaining a flush (all 5 cards in the hand are in the same suit)?

(c) What is the probability of obtaining three but not four Aces?

Solution

(a) $\binom{52}{5} = \frac{52!}{5!\,47!} = 2{,}598{,}960$ different 5-card hands.

(b) Let us first find the number of flushes. For a fixed suit of 13 cards, the number of ways of picking five cards is $\binom{13}{5}$. There are four suits. So

$$\#\text{ of flushes} = 4\binom{13}{5} = 5148$$

$$\text{Prob (5-card hand is a flush)} = \frac{4\binom{13}{5}}{\binom{52}{5}} = \frac{5148}{2{,}598{,}960} = 0.00198 \sim 0.2\%.$$

Answer

(c) We wish to count the number of hands with exactly three Aces. There are four Aces. Suppose we first pick three out of those four Aces. There are $\binom{4}{3} = 4$ ways of doing this. Having selected our three Aces, we need to pick two more cards for a hand, and these two cards must not contain the remaining Ace. In other words we have to pick two cards from 48 cards. There are $\binom{48}{2}$ ways of doing this. Putting it together we have:

$$\begin{aligned}\#\text{ of hands with exactly three Aces} &= \binom{4}{3}\binom{48}{2} \\ &= 4 \times 1128 \\ &= 4512.\end{aligned}$$

Therefore

$$\begin{aligned}\text{Prob (5-hand card has exactly three Aces)} &= \frac{4512}{2{,}598{,}960} \\ &= 0.00174.\end{aligned}$$

5. (Forming Committees) A committee of k people is to be chosen from a set of seven women and four men. How many ways are there to form the committee if

(a) The committee consists of three women and two men?

(b) The committee can be any positive size but must have equal number of women and men?

(c) The committee has four people and one of them must be Mr. Baggins

(d) The committee has four people and at least two are women?

(e) The committee has four people, two of each sex, and Mr. and Mrs Baggins cannot both be in the committee?

Solution

(a) Clearly the answer is $\binom{7}{3}\binom{4}{2} = \frac{7!}{3!\,4!}\ \frac{4!}{2!\,2!} = \frac{7!}{3!\,4}$

$= 7 \times 6 \times 5$

$= \boxed{210 \text{ ways}}$

(b) Since there are only four men (but seven women) the largest possible committee has four women and four men. So fix k between 1 and 4 $(1 \le k \le 4)$ and let us count the number of ways of forming a committee with k women and k men. The answer is clearly $\binom{7}{k}\binom{4}{k}$. The number of committees satisfying our requirements is the sum of these numbers as k ranges from 1 to 4:

$$\sum_{k=1}^{4} \binom{7}{k}\binom{4}{k} = \binom{7}{1}\binom{4}{1} + \binom{7}{2}\binom{4}{2} + \binom{7}{3}\binom{4}{3} + \binom{7}{4}\binom{4}{4}$$

$$= 7 \times 4 + \frac{7 \times 6}{2} \cdot \frac{4 \times 3}{2} + \frac{7 \times 6 \times 5}{3 \times 2} \cdot 4 + \frac{7 \times 6 \times 5}{3 \times 2} \cdot 1$$

$$= 28 + (21)(6) + (35)(4) + 35$$

$$= 28 + 126 + 175$$

$= 329$ ways.

(c) Out of the 11 people (seven women and four men) we have to form a committee of four persons, and one person has to be Mr. Baggins. The remaining three places in the committee have to be filled from the remaining 10 persons. Thus

\# of ways of forming a comittee of four with Mr. Baggins in committee

$$= \binom{10}{3}$$

$$= \frac{(10)(9)(8)}{(3)(2)}$$

$$= 120 \text{ ways}$$

(d) There are three possibilities for the committee composition:

(i) 2 women and 2 men

(ii) 3 women and 1 man

(iii) 4 women and no man.

For case (i) the count is $\binom{7}{2}\binom{4}{2} = \frac{7\times 6}{2} \cdot \frac{4\times 3}{2} = 126$

For case (ii) the count is $\binom{7}{3}\binom{4}{1} = \frac{7\times 6\times 5}{3\times 2} \cdot 4 = 140$

For case (iii) the count is $\binom{7}{4}\cdot\binom{4}{0} = \frac{7\times 6\times 5}{3\times 2} \cdot 1 = 35$

Adding we get:

\# of ways of forming committee as required $= 126 + 140 + 35$ ways

$$= 301 \text{ ways}$$

(e) There are two possibilities

(i) Mr. Baggins is in the committee

(ii) Mr. Baggins is not in the committee.

In the first case, Mrs. Baggins cannot be in the committee. We have to pick the remaining members of the committee by picking one man from the remaining three and two women from the remaining six (since Mrs. Baggins cannot be picked). This gives a count of

$$\binom{3}{1}\binom{6}{2} = 3 \times \frac{6!}{2!\,4!} = 3 \times \frac{6 \times 5}{2} = 45 \text{ ways}$$

If Mr. Baggins is not in the committee, there is no restriction on which women can be in the committee, and the two women members can be chosen in $\binom{7}{2}$ ways. The two men on the committee are chosen from a pool of three men (since Mr. Baggins is not in the committee). This means the men can be chosen in $\binom{3}{2} = 3$ ways. This gives us a count of

$$\binom{7}{2}(3) = \frac{7 \times 6}{2} \cdot 3 = 63 \text{ ways.}$$

Adding we get

# of ways of forming a committee as required = 45 + 63 ways

$= 108$ ways

Other random examples

I. Suppose there are 12 people in a room. What is the probability that no two of them have the same birthday. (Assume nobody was born on Feb 29, and that all other dates are equally likely.)

Soln: Let $A_1, A_2, \ldots, A_{12}$ be the persons in the room and $B_1, B_2, \ldots, B_{12}$ be their birthdates.

There are clearly 365^{12} possibilities for the sequence $B_1 B_2 \ldots B_{12}$. We wish to count the possibilities that the B_i's are all distinct, i.e., no two are the same.

There are 365 possibilities for B_1.

Once B_1 is fixed, there are 364 possibilities for B_2.

$\vdots$

Once $B_1, B_2, \ldots, B_{11}$ are fixed there are 354 possibilities for B_{12}.

Thus

\# of possibilities for B_i's to be all distinct $= 365 \times 364 \times 363 \times \cdots \times 354$

$$\boxed{\text{Prob}(\text{all } B_i\text{'s distinct}) = \frac{365 \times 364 \times 363 \times \cdots \times 357 \times 356 \times 355 \times 354}{(365)^{12}}}$$

II What fraction of all arrangements of INSTRUCTOR have three consecutive vowels?

Soln: The vowels are I, U, and O, and the remaining letters are N, S, T, T, R, R, C. The consonants are five types (N, S, T, R, C) with one of type 1, one of type 2, two of type 3, two of type 4, one of type 5, and there are seven of them. To this we can add an extra symbol X to stand for three consecutive vowels.

The symbols N, S, T, T, R, R, C, X can be arranged in $\binom{8}{2}\binom{6}{2}\cdot 4!$ ways — $\binom{8}{2}$ ways of picking two places for T, $\binom{6}{2}$ ways of picking two of the remaining spots for R, and $4!$ ways of arranging N, S, C, and X. Once we have such an arrangement, we have $3!$ number of ways of expanding X into an arrangement of I, U, O.

Thus the answer is:

$$3!\binom{8}{2}\binom{6}{2}\cdot 4! = 3!\,\frac{8!}{2!\,2!} = 60480.$$

Same argument as above.

The total # of arrangements $= \binom{10}{2}\binom{8}{2}\cdot 6! = \dfrac{10!}{2!\,2!}$

Proportion $= \dfrac{3!\,8!}{2!\,2!}\cdot\dfrac{2!\,2!}{10!} = \dfrac{3!\,8!}{10!}$ ← Answer.

The numbers $8!/2!2!$ and $10!/2!2!$ are denoted $P(8; 2, 2, 1, 1, 1, 1)$ and $P(10; 2, 2, 1, 1, 1, 1)$ respectively and their meaning is given in the next section.

Section 5.3 Arrangements and selections with repetition

Examples

I. How many ways are there to arrange the nine letters c, o, m, m, i, t, t, e, e?

Solution: We have nine places

□ □ □ □ □ □ □ □ □.

Pick two places – there are $\binom{9}{2}$ ways of doing this – and place the two m's in them. Now there are seven places left. Pick two from these — there are $\binom{7}{2}$ ways of doing this — and place the two t's in them. Now there are five places left. Pick two — there are $\binom{5}{2}$ ways of doing this — and place the two e's in them. In the remaining three places place the letters c, o, i and there are $3!$ ways of doing this. We can break down this last step into three. First pick one place — there are $\binom{3}{1}$ ways of doing this — and place c there. Of the remaining two places pick one — there are $\binom{2}{1}$ ways of doing this — and place o there. Finally in the last spot place i.

We therefore get

$$\#\text{ of ways of arranging } c,o,m,m,i,t,t,e,e = \binom{9}{2}\binom{7}{2}\binom{5}{2}\binom{3}{1}\binom{2}{1}\binom{1}{1}$$

obtained via straightforward cancellation

$$= \frac{9!}{2!\,2!\,2!\,1!\,1!\,1!}$$

This can be generalised to give:

Theorem 1: If there are n objects, with r_1 of type 1, r_2 of type 2, ..., and r_m of type m, where $r_1+r_2+\ldots+r_m=n$, then the number of arrangements of these objects, denoted $P(n; r_1, r_2, \ldots, r_m)$, is

$$\boxed{P(n; r_1, r_2, \ldots, r_m) = \binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\cdots\binom{n-r_1-r_2-\cdots-r_{m-1}}{r_m} = \frac{n!}{r_1!\,r_2!\cdots r_m!}}$$

Proof :

Note that the last factor, namely

$$\binom{n-r_1-r_2-\dots-r_{m-1}}{r_m}$$

is 1, because $n-r_1-r_2-\dots-r_{m-1}=r_m$ and $\binom{r_m}{r_m}=1$.

The theorem is clearly true when $m=1$.

Suppose it is true for $m=k$. We will prove it be true for $m=k+1$. Let us group the last two types of objects, i.e., type k and type $k+1$, into one large type. Now we have k types of objects, with type i having s_i number of objects where

$$s_i = r_i \quad \text{if} \quad 1 \le i \le k-1,$$

and

$$s_k = r_k + r_{k+1}.$$

By induction hypothesis the number of arrangements for this grouping is

$$P(n; s_1, s_2, \dots, s_k) = \binom{n}{s_1}\binom{n-s_1}{s_2}\cdots\binom{n-s_1-s_2-\dots-s_{k-2}}{s_{k-1}}\underbrace{\binom{n-s_1-s_2-\dots-s_{k-1}}{s_k}}_{\text{equal to } 1}$$

$$= \binom{n}{s_1}\binom{n-s_1}{s_2}\cdots\binom{n-s_1-s_2-\dots-s_{k-2}}{s_{k-1}}$$

$$= \binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_1-r_2-\dots-r_{k-2}}{r_{k-1}}$$

since $s_i = r_i$ for $1 \le i \le k-1$.

Once we have such an arrangement we look at the places occupied by the last type in the new grouping of types.

There are $r_k + r_{k+1}$ places. From these pick r_k spots and fill them with objects of the original type k. There are $\binom{r_k + r_{k+1}}{r_k}$ ways of finding r_k places from $r_k + r_{k+1}$ places. Now note that $r_k + r_{k+1} = n - r_1 - r_2 - \dots - r_{k-1}$, and so

$$\binom{r_1 + r_{k+1}}{r_k} = \binom{n - r_1 - r_2 - \dots - r_{k-1}}{r_k}.$$

So the number of arrangements of the original types is

$$P(n; s_1, \dots, s_k) \cdot \binom{n - r_1 - r_2 - \dots - r_{k-1}}{r_k} = \binom{n}{r_1}\binom{n}{r_2} \cdots \binom{n - r_1 - \dots - r_{k-1}}{r_k}$$

$$= \binom{n}{r_1}\binom{n}{r_2} \cdots \binom{n - r_1 - \dots - r_{k-1}}{r_k} \underbrace{\binom{n - r_1 - \dots - r_k}{r_{k+1}}}_{\text{Recall, this is 1.}}$$

$$= P(n; r_1, \dots, r_k, r_{k+1})$$

Thus by induction we are done.

Example: Suppose we have six objects A, B, C, D, E, F and they have to be put into 4 boxes K, L, M, N such that K contains two objects, L and M contain one object, and N contains two objects. How many ways can we do this?

Solution:

There are (at least) two ways of solving this problem.

Method 1: Pick two objects from A, B, C, D, E, F and put them in K. There are $\binom{6}{2}$ ways of doing this. Of the remaining four objects, pick one and put it in L. There are $\binom{4}{1}$ ways of doing this. Of the remaining three objects pick one and put it in M. There are $\binom{3}{1}$ ways of doing this. The last two objects go into N. There is of course only one way of doing this, and note $\binom{2}{2}=1$.

So the answer is:

$$\binom{6}{4}\binom{4}{1}\binom{3}{1}\binom{2}{2} = P(6;2,1,1,2) = \frac{6!}{2!\,2!} = 180 \longleftarrow \text{Answer}$$

The second method involves a different way of thinking.

<u>Method 2</u>: The trick is to have two copies K and two copies of N; thus we have 6 boxes: K, K, L, M, N, N. There are $P(6;2,1,1,2)$ ways of arranging this. Suppose for example LKNNKM is an arrangement. Then put A into L, B into K, C into N, D into N, E into K, F into M. Then K contains B and E, L contains A, M contains F, and N contains C and D.

$$\left.\begin{array}{l} A \longrightarrow L \\ B \longrightarrow K \\ C \longrightarrow N \\ D \longrightarrow N \\ E \longrightarrow K \\ F \longrightarrow M \end{array}\right\} \text{Allocation of objects for the arrangement LKNNKM}$$

B, E	A	F	C, D
K	L	M	N

Similarly given any arrangement of K, K, L, M, N, N, we can find a way of distributing A, B, C, D, E, F in K, L, M, N according to the required rules, and conversely every such distribution gives us an arrangement of K, K, L, M, N, N. For example A, F in K, E in L, C in M, and B, D in N corresponds to the arrangement KNMNLK. Clearly there are $P(6; 2,1,1,2)$ ways of arranging K, K, L, M, N, N. Thus the answer to the question is, as before, $P(6; 2,2,1,2)$.

$$P(6; 2,1,1,2) = \frac{6!}{2!\,1!\,1!\,2!} = 180 \longleftarrow \text{Answer.}$$

* **IMPORTANT**: Read Example 4 from Section 5.4.