Mar 1, 2018

Lecture 14

Consider three objects a, b, and c. The various arrangements

(*) ——— abc, acb, cba, bac, bca, cab

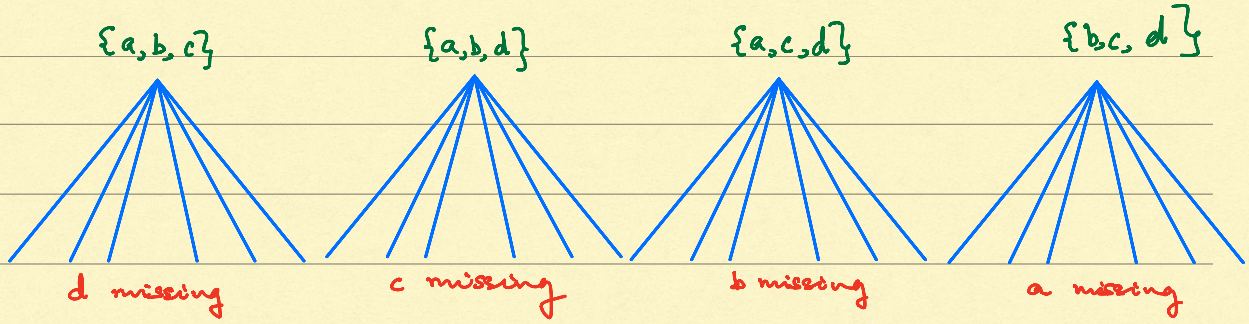
are called <u>permutations</u> of a, b, and c.

In a little greater generality, suppose a, b, c, d are five distinct objects. Arrangements of subsets of three objects are

$\underbrace{abc, acb, \ldots, cab}_{\text{no } d}, \underbrace{abd, adb, \ldots, dab}_{\text{no } c}, \underbrace{acd, adc, \ldots, dac}_{\text{no } b}, \underbrace{bcd, bdc, \ldots, dbc}_{\text{no } a}.$

There are 24 such arrangements. The are called the <u>3-permutations</u> of the 4 distinct objects a, b, c, d.

These 24 3-permutations can be counted in the following way. First pick 3 objects from a, b, c, d and then arrange the three objects. There are 4 ways of picking 3 objects from a, b, c, d, because picking 3 objects is the same as leaving out one objects and there are 4 ways of doing that. Once that is done, there are 6 ways of arranging them as we saw in (*) above



$6 \times 4 = 24.$

A choice of 3 objects from a, b, c, d, i.e., a choice of a subset of three elements from the set $\{a, b, c, d\}$, is called a <u>3-combination</u> of the four distinct objects a, b, c, d. As we have seen, there are four 3-combinations of a, b, c, d.

This can be generalised.

Definition: A <u>permutation</u> of n distinct objects is an arrangement, an ordering, of the n objects. An <u>r-permutation</u> of n distinct objects is an arrangement of using r of the n objects. An <u>r-combination</u> of n distinct objects is an unordered selection, or <u>subset</u>, of r out of the n objects.

Notations:

$P(n,r)$ = # of <u>r-permutations</u> of n objects.

$C(n,r)$ or $\binom{n}{r}$ = # of <u>r-combinations</u> of n objects.

Both these symbols are used for this.

Formulas for $P(n,r)$ and $\binom{n}{r}$:

First let us calculate the number of ways of arranging r objects $a_1, a_2, \ldots, a_r$. Let $x_1 x_2 \ldots x_r$ be an arrangement of $\{a_1, a_2, \ldots, a_r\}$.

There are r ways of picking x_1 from $\{a_1, \dots, a_r\}$.

Having picked x_1, there are $r-1$ ways of picking x_2 from the remaining a_i's.

$\vdots$

If $x_1, x_2, \dots, x_k$ have been picked $(1 \le k \le r-1)$ then there are $r-k$ ways of picking x_{k+1} from the remaining a_i's.

It follows that there are

$$r(r-1)\cdots(2)(1)$$

ways of arranging $\{a_1, a_2, \dots, a_r\}$ as $x_1 x_2 \dots x_r$.

<u>Notation</u>

$$r! = r(r-1)\cdots(2)(1)$$

r "factorial".

Next let us work out $P(n,r)$ using the same reasoning. Now we have to count the number of arrangements $x_1 x_2 \dots x_r$ from n objects $\{a_1, a_2, \dots, a_n\}$ ← n objects.

Now there are n ways of picking x_1. Having picked x_1, there are $n-1$ ways of picking x_2. We continue until we pick x_r and then stop. Therefore

$$P(n,r) = n(n-1)\cdots(n-i)\cdots(n-r+1)$$

# of ways of picking x_1.

# of ways of picking x_2 once x_1 has been picked

# of ways of picking x_{i+1} once $x_1, x_2, \dots, x_i$ have been picked

# of ways of picking x_r once $x_1, x_2, \dots, x_{r-1}$ have been picked.

So

$$P(n,r) = n(n-1)\cdots(n-r+1)$$

$$= \frac{n(n-1)\cdots(n-r+1)(n-r)(n-r-1)\cdots(2)(1)}{(n-r)(n-r-1)\cdots(2)(1)}$$

$$= \frac{n!}{(n-r)!}$$

Thus

$$\boxed{P(n,r) = \frac{n!}{(n-r)!}}$$

It is also clear that $P(n,r)$ can be computed another way. First pick r objects from n objects in an unordered. There are $\binom{n}{r}$ ways of doing this. Once we pick r objects we can arrange them in $r!$ ways. It follows that

$$\boxed{P(n,r) = r!\binom{n}{r}}$$

Thus

$$\binom{n}{r} = \frac{P(n,r)}{r!}$$

i.e.,

$$\boxed{\binom{n}{r} = \frac{n!}{r!\,(n-r)!}}$$

Summary of formulas

$$P(n,r) = \frac{n!}{(n-r)!}$$

$$P(n,r) = r!\binom{n}{r}$$

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Note that from the 3rd formula we have $\binom{n}{r} = \binom{n}{n-r}$. This can also been seen combinatorially — a choice of r objects from n is a choice of the remaining $n-r$ objects from n objects.

Examples

1. (Ranking wizards) How many ways are there to rank n candidates for the job of chief wizard? If the ranking is made at random (each ranking is equally likely), what is the probability that the fifth candidate, Gandalf, is in second place?

Solution

Candidates: $C_1, C_2, C_3, C_4, C_5, \ldots, C_{n-1}, C_n$

Gandalf

Ranking: $X_1, X_2, X_3, \ldots, X_n$

$X_1 X_2 \ldots X_n$ a permutation of $\{C_1, \ldots, C_n\}$

Since there are $n!$ permutations of $\{C_1, \ldots, C_n\}$:

Total # of possible rankings $= n!$

Now consider all rankings with the 2^{nd} place occupied by C_5, i.e., by Gandalf.

$$X_1 \, X_2 \, X_3 \, \ldots \, X_n$$

Equal to C_5,
i.e. X_2 is Gandalf.

This is the same as all permutations of the form

$$X_1 \, C_5 \, X_3 \, X_4 \, \ldots \, X_n .$$

So really we are counting arrangements of $\{C_1, \ldots, C_n\} - \{C_5\}$, i.e., of $n-1$ objects, and the number of such arrangements is $(n-1)!$.

# of rankings with Gandalf in the 2^{nd} place $= (n-1)!$

Since all rankings are equally likely,

$$\text{Prob (Gandalf second)} = \frac{\text{\# of rankings with Gandalf } 2^{nd}}{\text{total \# of rankings}}$$

$$= \frac{(n-1)!}{n!}$$

$$= \frac{1}{n}$$ ← Answer

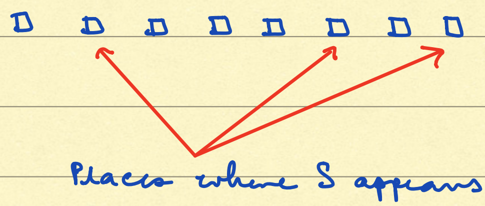
We point out that this is intuitively the correct answer for the probability of Gandalf appearing in any rank from 1 through n should be the same, and hence should be $1/n$.

2. (Arrangements with Repeated Letters) How many ways are there to arrange the eight letters in the word BUSINESS? In how many of these arrangements do the three Ss appear consecutively?

Solution

There are eight spots.

The Ss appear in 3 of them, and there is no way of distinguishing them from each other. Pick the places where the Ss will appear.



There are $\binom{8}{3}$ ways of picking 3 places out of 8. Put Ss in these places once they have been picked. Once 3 places have been picked the remaining five letters B, U, I, N, E have to be put into the remaining five spots. There are 5! ways of doing this. It follows that

$$\#\text{ of arrangements of the letters in BUSINESS} = \binom{8}{3}5!$$

$$= \frac{8!}{3!\,5!}\,5! = \frac{8!}{3!}$$

$$= (8)(7)(6)(5)(4) = 6720$$

Notice that the answer $\frac{8!}{3!}$ is the same as $P(8,5)$. There is a simpler way of solving the problem. Suppose we have a 5-permutation of $\{1,2,3,4,5,6,7,8\}$, say $x_1 x_2 x_3 x_4 x_5$. Then we put B in the x_1-th spot, U in the x_2 th spot, I in the x_3 th spot, N in the x_4-th spot, and E in the x_5 th spot, and Ss in the remaining. Conversely given an arrangement of B, U, S, I, N, E, S, S we call the spot B is in as x_1, the one U is in as x_2, etc to get an 5-permutation of $\{1,2,3,4,5,6,7,8\}$.

So either way the answer is $P(8,5)$, i.e., 6720.

For the second part, treat the three consecutive Ss, i.e., SSS, as a single letter. So we now have to arrange the six objects B, U, I, N, E, and SSS. There are $6!$ ways of doing this. Since $6! = 720$, the answer is 720.

<u>Answers</u>: 6720 and 720

or $P(8,5)$ and $6!$ ← More conceptual.

This way of doing the problem was not presented in class because of lack of time. However, do read it for it is an interesting way of looking at the problem.

Example 2 continued: Consider again the word BUSINESS. We saw that there were $P(8,5)$ number of arrangements of the eight letters in the word. One could ask how many arrangements have I occurring somewhere before E? One could also ask - how many such arrangements (I before E) have the three S's grouped consecutively?

For the first problem, pick 3 spots for the S's. There are $\binom{8}{3}$ such spots. Of the remaining 5 spots pick 2 spots and place I in the spot which occurs first and E in the one that occurs second. There are $\binom{5}{2}$ ways of picking these two spots. This leaves 3 spots and place the remaining letters B, U, N in these three spots. There are $3!$ ways of doing this. Thus the answer in this case is:

$$\binom{8}{3}\binom{5}{2}\cdot 3! = \frac{8!}{3!\,5!}\cdot\frac{5!}{3!\,2!}\cdot 3! = \frac{8!}{3!\,2!}$$

Answer

Another way: Pick 2 spots for I and E and place I in the 1st, E in the 2nd. There $\binom{8}{2}$ ways of picking 2 spots out of 8. Of the remaining 6, find 3 to place B, U, N in various arrangements. This can be done in $P(6,3)$ ways. The remaining three spots are filled with S's. This gives us

$$\binom{8}{2}P(6,3) = \frac{8!}{2!\,6!}\cdot\frac{6!}{3!} = \frac{8!}{2!\,3!}$$

as the answer. This agrees with the older answer.

Now for the second question.

We want the number of arrangements of B, U, S, I, N, E, S, S with I before E and the three S's bunched together.
We again regard SSS as a single letter. Thus we have six "letters": B, U, I, N, E, SSS. This gives us six spots:

$\square \ \square \ \square \ \square \ \square \ \square$

Pick two and place I in the first and E in the second. There are $\binom{6}{2}$ ways of doing this. Now place the remaining four objects B, U, N, SSS in the remaining four spots. There are $4!$ ways of doing this. The answer is

$$\binom{6}{2} \cdot 4! = \frac{6!}{2! \cdot 4!} \cdot 4! = \frac{6!}{2!} = 360$$

← Answer