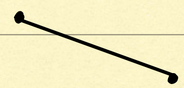
Feb 15, 2018

# Lecture 13

Example on degrees of regions:

For the problem of showing that a connected planar graph with eight vertices and 13 edges has a triangle as one of its regions one needs the notion of a degree of region which we have done earlier. Recall that $R_1, \ldots, R_n$ are the regions cut out by a planar depiction of a planar graph $G$, we have $\sum_{i=1}^{n} \deg(R_i) = 2(\#\text{ of edges in } G)$. It is not possible for a region to have degree 1. Also, if a planar depiction of a planar graph $G$ has a region $R$ of degree 2, then $G$ must be a graph with exactly one edge.



Here is a related problem:

Suppose $G$ is a connected planar graph with 8 vertices and 12 edges. Show that every planar depiction of $G$ has at least two regions of degree $\leq 4$.

Solution: From Euler's formula we see that the number of regions defined by any planar depiction of $G$ is 6. Suppose $R_1, R_2, R_3, R_4, R_5$ and $R_6$ are the regions. We first show that at least one of the regions has degree $\leq 4$.

Suppose this is not so. Then

$$\deg R_i \geq 5$$

for every $i$.

Then

$$24 = 2(\#\text{ of edges of } G) = \deg R_1 + \deg R_2 + \deg R_3 + \deg R_4 + \deg R_5 + \deg R_6$$
$$\geq 5 + 5 + 5 + 5 + 5 + 5$$
$$= 30.$$

In other words $24 \geq 30$, which is impossible. Therefore there is at least one region with degree $\leq 4$.

Next we show that there must be at least two regions with $\deg \leq 4$. Suppose not. Then there is exactly one region with $\deg \leq 4$. Let this region be $R_1$. Then $\deg R_1 = 3$ or $4$, and $\deg R_i \geq 5$ for $i = 2, \ldots, 6$. Again we have

$$24 = \deg R_1 + \sum_{i=2}^{6} \deg R_i$$

$$\geq \deg R_1 + (5)(5)$$
$$= \deg R_1 + 25$$

which means $\deg R_1 \leq -1$, an impossibility. Thus we must have at least two regions with $\deg \leq 4$.

<u>Note</u>: For $G$ as in this problem, this means there is at least one <u>bounded</u> region with $\deg \leq 4$. This is so, because there is only one unbounded region defined by a planar depiction of a planar graph.

<u>Two important network flow equations</u>:

Let $N = (V, E, k)$ be directed network, $P$ a subset of $V$, $a, z$ two vertices, and $f$ an $a$-$z$ flow for $N$. $\bar{P}$ denotes the complement of $P$, and $(P, \bar{P})$ is the cut consisting of edges from $P$ to $\bar{P}$.

① If $P$ does not contain $a$ or $z$ then

$$\sum_{e \in (P,\overline{P})} f(e) = \sum_{e \in (\overline{P},P)} f(e)$$

② If $(P,\overline{P})$ is an $a$–$z$ cut (i.e., $a \in P$ and $z \in \overline{P}$), then

$$\sum_{e \in (P,\overline{P})} f(e) = |f| + \sum_{e \in (\overline{P},P)} f(e)$$

The proofs have been given in the notes for Lecture. There is also a different and very elegant proof of the second equation in the textbook.

# Chapter 5

## Section 5.1

### Examples

1. Two dice are rolled, one green and one red. Each die has faces numbered 1 through 6.

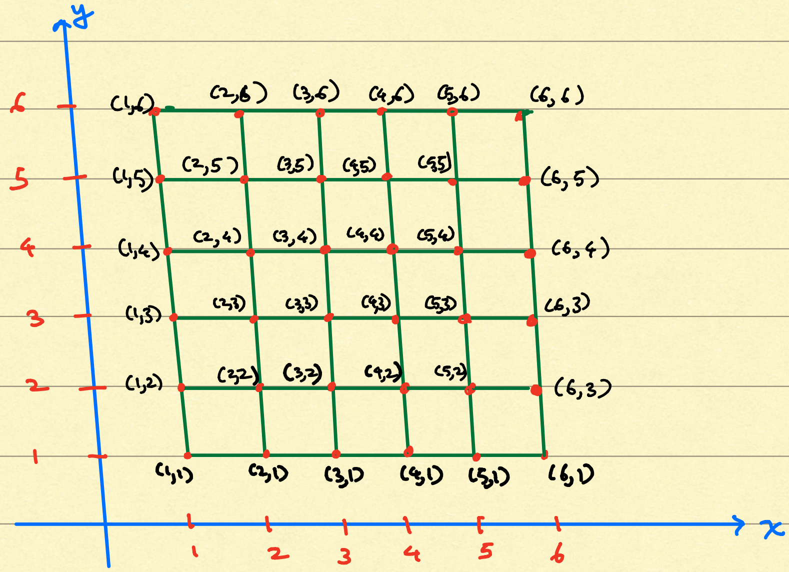
   (a) How many outcomes for the procedure are there?

   (b) What is the probability that there are no doubles (not the same value on both dice)?

**Solution:**

(a) The various outcomes can be represented by ordered pairs $(x, y)$ where $x$ is the number showing up on the green die and $y$ the one on the red. Note that $1 \le x \le 6$ and $1 \le y \le 6$. There are $6 \times 6 = 36$ such ordered pairs. (See picture below.)

Thirty six ordered pairs.




Thus # of outcomes = 36.

(b) "Doubles" amount to ordered pairs of the form $(x,x)$, i.e. the elements in the rectangular grid above which lie along the diagonal in the rectangle from $(1,1)$ to $(6,6)$. These are six elements. So the number of outcomes without doubles is $36-6=30$.

There is another way of thinking about this: If no doubles are allowed, then for each fixed $x$, there are five possible $y$'s. So the number of such outcomes is $6 \times 5 = 30$.

The probability of no doubles occurring is $= \dfrac{\text{\# of outcomes without doubles}}{\text{Total \# of outcomes}}$

$$= \frac{30}{36}$$

$$= \frac{5}{6}$$

Answer: 5/6.

2. There are five different Spanish books, six different French books, and eight different Transylvanian books. How many ways are there to pick an (unordered) pair of two books not both in the same language.

Solution:

Any such choice of two books entails one language whose books have not been picked. There are therefore three broad possibilities: (i) Spanish books are not picked, (ii) French books are not picked, (iii) Transylvanian books are not picked.

In case (i) we have $6 \times 8 = 48$ possibilities.

In case (ii) we have $5 \times 8 = 40$ possibities.

In case (iii) we have $6 \times 5 = 30$ possibities.

Thus:

The total # of ways of picking books according to the prescribed rules

$$= 48 + 40 + 30$$

$$= 118$$

No Spanish book

Six ways of choosing French books. For each choice of French book, have eight choices of Transylvanian books. So $6 \times 8 = 48$ possibities.

48

No French book

Five ways of choosing a Spanish book, and for each choice, have eight choices of Transylvanian books. So $5 \times 8 = 40$ possibidities

40

No Transylvanian book

Five ways of picking a Spanish books, and for each choice, six ways of picking a French book. So $5 \times 6 = 30$ possibilities

30

Answer: 118

3. How many ways are there to form a three letter sequence using the letters a, b, c, d, e, f (a) with repetition of letters allowed? (b) without repetition of any letter? (c) without repetition and containing the letter e? (d) with repetition and containing e? Ans: (a) $6^3 = 216$; (b) 120; (c) 60; (d) 91

Solution:

Let $xyz$ be the three letter sequence formed from a, b, c, d, e, f.

(a) The answer is clearly $6 \times 6 \times 6 = 216$. Answer: 216

(b) There are six ways of picking $x$. Having picked $x$, there are five ways of picking $y$ so that $y \neq x$. Once $x$ and $y$ have been chosen so that $x \neq y$, there are four ways of picking $z$ so that $z \neq x$ and $z \neq y$. Thus there are $6 \times 5 \times 4 = 120$. Answer: 120

(c) One of $x$, $y$, or $z$ is e. Suppose $x = e$. Then there are five possibilities for $y$ and four possibilities for $z$. Thus we have $5 \times 4 = 20$ possibilities here. The same reasoning applies if $y$ or $z$ equals e.
Thus we have $20 + 20 + 20 = 60$. Answer: 60

(c) We have three mutually exclusive possibilities:

(i) $x = e$

(ii) $y = e$ but $x \neq e$.

(iii) $z = e$ but $x \neq e$ and $y \neq e$.

The count for case (i) is $6 \times 6 = 36$.

For (ii) the count is $5 \times 6 = 30$

For (iii) the count is $5 \times 5 = 25$

The # of possibilities $= 36 + 30 + 25 = 91$. Answer: 91

4. How many non-empty different collections can be formed from five (identical) apples and eight (identical) oranges?

Solution:

This is straight forward. A collection can be represented by an ordered pair $(x, y)$ with $x$ being the # of apples in the collection and $y$ the # of oranges. The possibilities for $x$ are $0, 1, 2, 3, 4, 5$ and that for $y$ is $0, 1, 2, 3, 4, 5, 6, 7, 8$. The pair $(0,0)$ represents the empty collection. The total number of non-empty collections is $(9 \times 6) - 1 = 54 - 1 = 53$. Answer: 53.