Feb 6, 2018

Lecture 10

In class today we completed the proof of

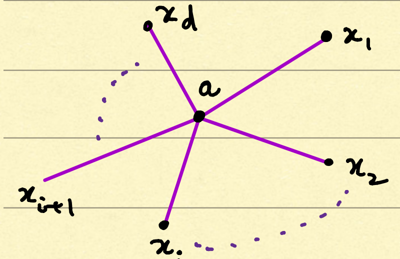
Theorem: Every planar graph can be 5-coloured.

The proof is given in the notes in Lecture 9. The technique of deleting a vertex (and all adjacent edges) and then re-introducing the vertex in the induction argument used for that proof can be repeated to prove the following:

Theorem: If the maximum degree of a vertex in a graph G is d, then G is $(d+1)$-colourable.

Proof:

The proof is by induction on the number of vertices n. G has at least $d+1$ vertices since it has a vertex of degree d.



Vertex a of degree d. There are $(d+1)$-vertices on display: $a, x_1, x_2, \ldots, x_d$.

If $n = d+1$, then clearly G is $(d+1)$-colourable. Now suppose $n > d+1$. Suppose also that the theorem is true for graphs with fewer than n vertices.

Let a be a vertex of G of degree d. Remove a and all edges incident on a to form a new graph G'. The graph G' has $n-1$ vertices. Further the maximum degree of G' is less than or equal to d. By our induction hypothesis, G' is $(d+1)$-colourable. The d vertices adjacent to a use up at most d colours. This leaves at least one colour unused by the vertices adjacent to a. Use this colour for a.

q.e.d.

Colouring theorems (without proofs):

The most famous colouring theorem is

The 4-colour theorem (Appel & Haken, 1976): Every planar graph can be 4-coloured.

(This means if G is planar, $\chi(G) \leq 4$).

This was the most important unsolved problem in graph theory for most of the 20th century. When Appel and Haken solved it, they had examined 1955 classes of graphical configurations, each of which involved numerous subcases. Needed immense, computer-generated, case by case, exhaustive analysis. Needless to say, we won't be giving a proof.

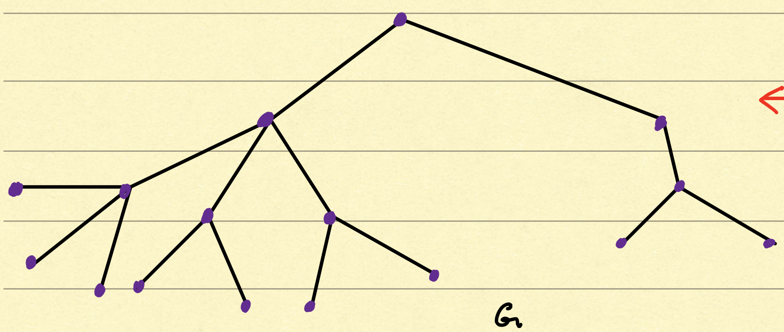
Theorem (Brooks, 1941): If the graph G is not an odd circuit or a complete graph, then $\chi(G) \leq d$, where d is the maximum degree of a vertex of G.

Instead of colouring vertices, we can colour edges so that no two edges with a common vertex have the same colour. One then has an obvious notion of an edge chromatic number of a graph G, namely the fewest number of colours required to colour the edges of G.

<u>Theorem (Vizing, 1964)</u>: If the maximum degree of a vertex in a graph G is d, then the edge chromatic number of G is either d or d+1.

Chapter 3 Trees and Searching

Consider the graph G below:



G

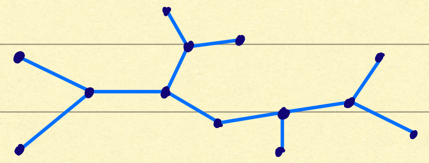
Such graphs are called TREES.

The important features are:

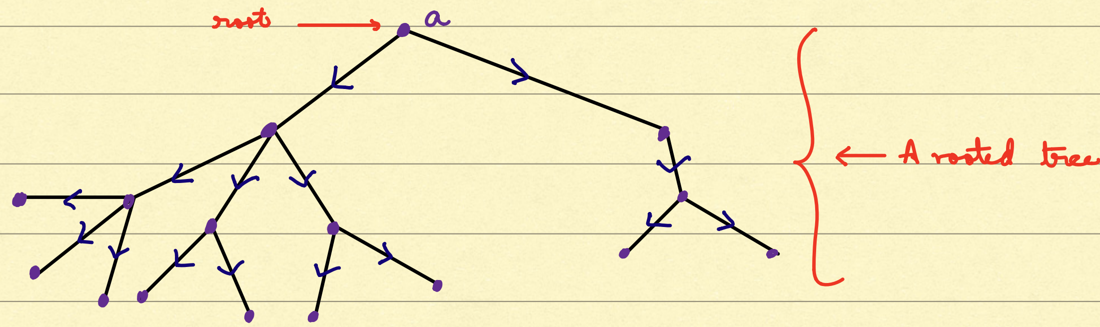
- G has no circuits.
- Given any two distinct vertices in G, there is a <u>unique path</u> joining them.
- If any edge is removed, the graph disconnects.

Such graphs are called <u>trees</u>.

Here is one more example of such a graph.



The following is an example of a directed graph with the property that there is a unique directed path from a certain vertex, called the root, and any other vertex. Note that the underlying undirected graph is the first graph we considered.



The following theorem characterizes these kinds of undirected graphs.

Theorem: Let T be a connected undirected graph. Then the following statements are equivalent.

(a) T has no circuits

(b) There is a unique path between any pair of distinct vertices x, y in T.

(c) T is minimally connected in the sense that the removal of any edge in T will disconnect T.

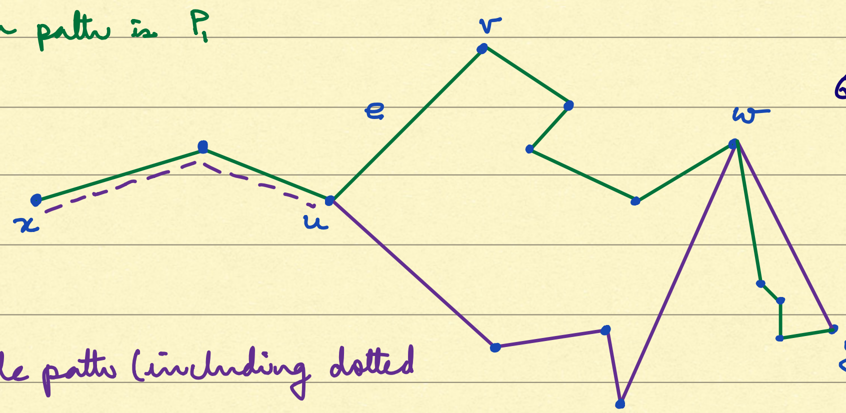
Proof:

$(a) \Longrightarrow (c)$: Let us assume (a) is true. We have to show (c) is true. Suppose removal of an edge (α, β) does not disconnect T. Then there is a path P in the graph $T - (\alpha, \beta)$ which connects α and β. This means P plus the edge (α, β) is a circuit, violating (a).

$(c) \Longrightarrow (b)$: Assume (c). Suppose there are two distinct paths P_1 and P_2 connecting x and y. Starting from x, let $e = (u, v)$ be the first edge on P_1 that is not on P_2. Start from u, and follow P_2 until y. Continue following P_2 until the first vertex also on P_1 is encountered; call this vertex w. Now return from w to v following P_1. This procedure gives us a path Q from u to v in $T - e$, i.e., $T - e$ is connected. This violates (c).

$(b) \Longrightarrow (a)$: If T has property (b) then it cannot have circuits, for any circuit, for the edges of a circuit provide two different paths joining any two vertices that lie on the circuit.

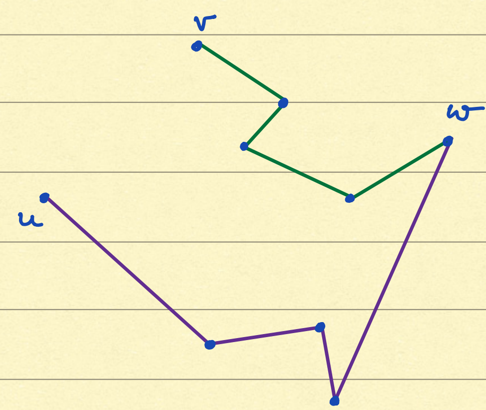
Green path is P_1



Q is the path from u to w along the purple path followed by w to v along the green path.

Purple path (including dotted purple path) is P_2.

Below is the path Q $(Q \neq e)$



Definition: A tree is a connected graph T satisfying any of the equivalent conditions of the theorem. A rooted tree (or a directed tree) is a directed graph T together with a special vertex called the root such that there is a unique directed path from the root to any other vertex in T and no directed edge ending at the root.

Note: If T is a rooted tree with root a, then for any vertex v different from a, there is a unique incoming vertex $(\vec{u}, v)$ at v. First, since there is directed path from a to v, v has an incoming edge. To see this is the only one, we argue as follows. Suppose $(\vec{w}, v)$ is another incoming edge at v. Let P be the unique directed path from a to u and Q the unique directed path from a to w. Then $P + (\vec{u}, v)$ and $Q + (\vec{w}, v)$ are directed paths from a to v. So they must be equal. This means $P = Q$

and $(\vec{u},v) = (\vec{w},v)$. This means $u=w$ and we have only one incoming edge at v.

Thus, in a rooted tree, the root is the only vertex without an incoming edge. All other vertices have a unique incoming edge.

Definition: Let T be a rooted tree with root a. If $(\vec{u},v)$ is a directed edge in T, then u is called the parent of v. The children of a vertex x are vertices z with a directed edge from x to z. Two vertices with the same parent are siblings. If there is a directed path from a vertex p to a vertex q then p is an ancestor of q and q is a descendent of p.

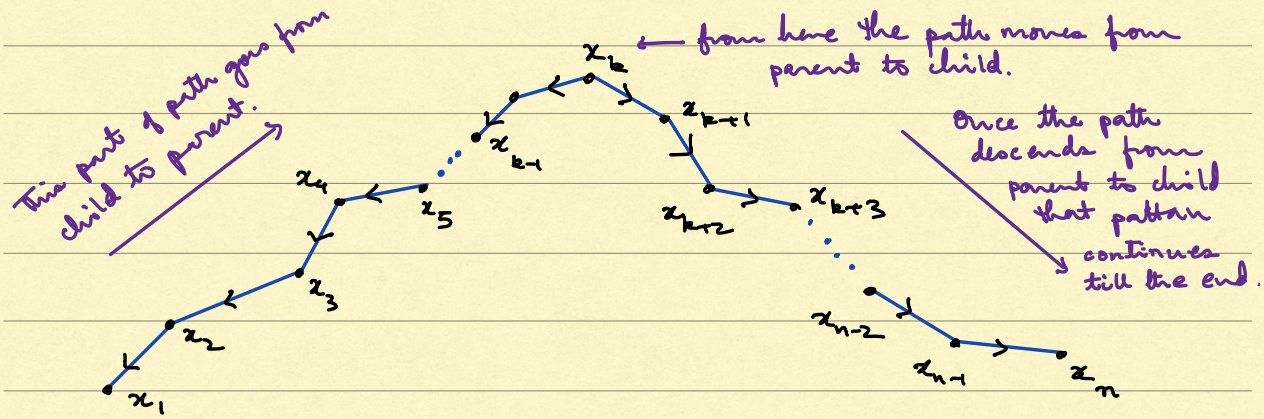
Some observations: Let T be a rooted tree.

- The root is an ancestor of all other vertices, and all vertices are descendents of the root.
- A vertex which is not a root has exactly one parent.
- The root has no parents or ancestors.
- If q is a descendent of p, there is a unique directed path from p to q.
- A vertex p is an ancestor of a vertex q if and only if the unique directed path from the root to q passes through p.
- Let G be the undirected graph obtained from T by

forgetting about the directions of the edges in T. If P is a path in G from x to y, say

$$P = x_1 - x_2 - \dots - x_n$$

For each i, x_{i+1} is either a parent of x_i or a child of x_i. If x_{i+1} is a child of x_i, then x_{i+2} must be a child of x_{i+1}, otherwise x_{i+1} has two parents, x_i and x_{i+2}, but we have observed that a vertex can have at most one parent. In T, the path P must look like this.



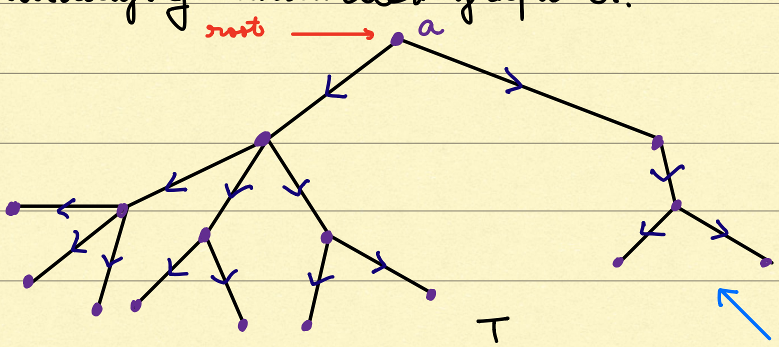
- Again, if G is the underlying undirected graph, and x is any vertex which is not the root a, then in G there is a unique path from a to x. From the previous observation, any path from a to x must have each vertex succeeded by a child, since a has no parent. This means the path can be regarded as a directed path in T. In which case, by definition, this path is unique.
- Let G be as above (obtained from T by forgetting about the

edge directions). Let $(\overrightarrow{x,y})$ be an edge in T. If we remove (x,y) from G, then G is disconnected. If not, there is a path P from the root a of T to x in $G-(x,y)$ and another path Q from a to y in $G-(x,y)$. This means $P+(x,y)$ and Q are two paths in G from a to y, and this contradicts the previous observation.

The last observation shows:

> If T is a rooted tree, then the underlying undirected graph is a tree.

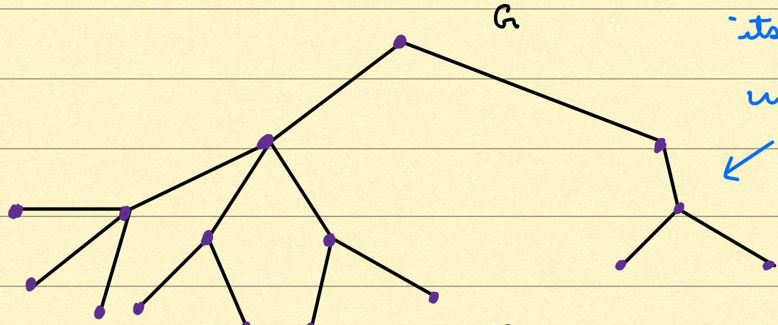
Here is a pictoral example of a rooted tree T and its underlying undirected graph G.



T

G

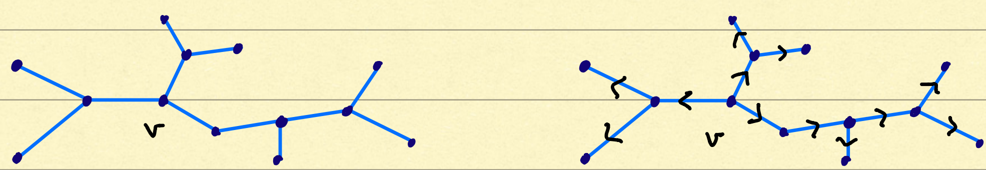
Rooted tree and its underlying undirected graph



G

Conversely if G is a tree, then we can make it a rooted tree by making any vertex we wish as a root. Here is how we do this. Suppose (x,y) is an edge in G. If the unique path from v to x does <u>not</u> pass through y, then give the edge the direction from x to y $x \longrightarrow y$. If the unique path from v to x passes through y, give the edge the direction from y to x $x \longleftarrow y$. In this way every edge is given a direction, and we have a directed graph T. It is easy to see that T is a rooted tree with root v.

> If G is an undirected tree and v is a vertex in G then G can be made into a rooted tree T in a unique way with v is its root.



$G \longrightarrow T$

Pick any vertex v in G. Can make G into a rooted tree T with root v as in the picture. Use the property that in a rooted tree, a vertex can have at most one incoming edge (exactly one if the vertex is not a root).

Theorem: A tree with n vertices has $n-1$ edges.

Proof:

Without loss of generality we may assume that the tree is a rooted tree. There are $n-1$ vertices which are not the root and each has a unique incoming edge. This accounts for all the edges. q.e.d.

Definition: Let T be a rooted tree. Vertices of T with no children are called leaves of T. Vertices with children are called internal vertices of T. If every internal vertex of T has m children, T is called an m-ary tree. If $m=2$, T is a binary tree.

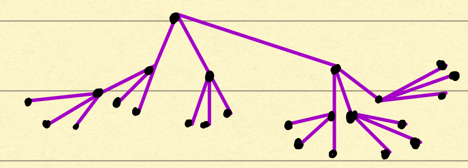
Theorem: Let T be an m-ary tree with n vertices of which i are internal. Then $n = mi+1$.

Proof: The internal vertices are exactly the vertices which are the parents of some vertex. The i internal edges have m children each, and since a vertex can have only one parent, this accounts mi children, and mi is the number of vertices which are children of some vertex. The root is the only vertex which is not the child of any vertex so adding that in we see that the total number of vertices is

$$n = mi+1.$$

q.e.d.

Suppose T is an m-ary tree with i internal vertices and l leaves. If n is the total number of vertices then



3-ary tree.

$l + i = n$.

On the other hand, from the previous theorem we have $n = mi + 1$.

We thus have two equations

$$\left.\begin{array}{l} mi+1 = n \\ l + i = n. \end{array}\right\} \text{———} \quad (*)$$

The system of equations $(*)$ can be re-arranged in a number of ways.

Corollary on bottom of p. 95 of the textbook. →

$$\left.\begin{array}{l} l = (m-1)i+1 \\ n = mi+1 \end{array}\right\} \text{———} \quad (a)$$

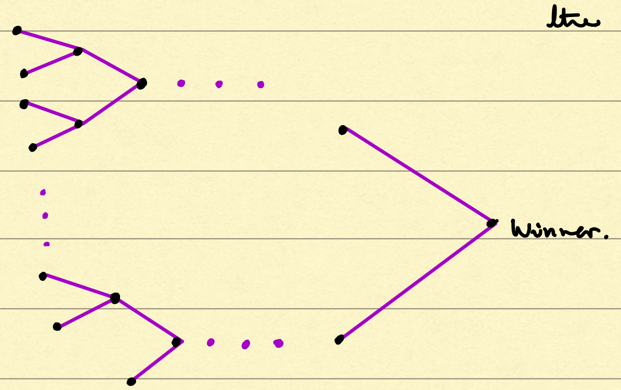
$$\left.\begin{array}{l} i = \dfrac{l-1}{m-1} \\ n = \dfrac{ml-1}{m-1} \end{array}\right\} \text{———} \quad (b)$$

$$\left.\begin{array}{l} i = \dfrac{n-1}{m} \\ l = \dfrac{(m-1)n+1}{m} \end{array}\right\} \text{———} \quad (c)$$

(a), (b), and (c) are systems of equations, each equivalent to the system $(*)$ given above.

One can use (a) to find l and n if i and m are given; (b) to find i and n if l and m are given; and (c) to find i and l if n and m are given.

Example: A tennis tournament can be modelled as a binary tree with the 1st round entrants as the leaves. The matches are the internal vertices.



Q: If 56 people sign up for a tennis tournament, how many matches will be played?

Solution: Here $l = 56$ and $m = 2$. Use the system of eqns in (b). We have to find i.

$$i = \frac{l-1}{m-1} = \frac{56-1}{2-1} = 55.$$

So 55 matches will be played.

Definitions: The height of a rooted tree is the length of the longest path from the root. The level number of a vertex x in a rooted tree is the length of the unique path from the root to x. A rooted tree of height h is called balanced if all leaves are at levels h and $h-1$.