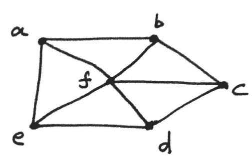
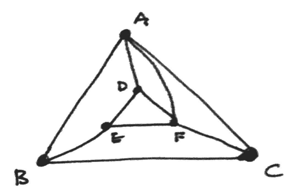
MAT 344 INTRODUCTION TO COMBINATORICS
SOLUTIONS OF THE MIDTERM TEST ON FEB 27, 2018

(1) (20 marks) Draw two non-isomorphic graphs with six vertices and 10 edges. Prove the two graphs you have drawn are not isomorphic.

Solution: Consider the following two graphs





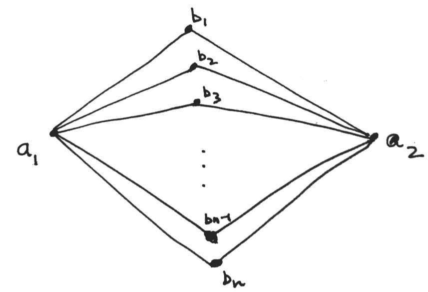
Let the graph on the left be denoted G and the one on the right G'. They both have six vertices and 10 edges. However, G has a vertex, namely f, which has degree 5, whereas none of the vertices of G' have degree 5. So G and G' cannot be isomorphic. (This in NOT the only solution. There may be many other pairs of non-isomorphic graphs with six vertices and 10 edges.) □

(2) Let K_n denote the complete graph with n vertices, and $K_{m,n}$ the complete bipartite graph with m vertices on the left, and n vertices on the right.

(a) (10 marks) Show that $K_{2,n}$ is planar for every $n \geq 1$.

(b) (10 marks) Show that K_5 is not planar by using the corollary to Euler's formula connecting the number of regions, edges, and vertices.

Solution:
(a) The following picture shows that $K_{2,n}$ is planar for every $n \geq 1$.



(b) The corollary states that $\mathbf{e} \leq 3\mathbf{v} - 6$ for any planar graph G where $\mathbf{e}$ is the number of edges of G and $\mathbf{v}$ is the number of vertices of G. If G is K_5 then $\mathbf{e} = 10$ and $\mathbf{v} = 5$. In this case $3\mathbf{v} - 6 = 15 - 6 = 9$, and hence $\mathbf{e} > 3\mathbf{v} - 6$, which means K_5 is not planar. □

(3) (20 marks) Suppose a graph G has all vertices of degree $\leq d$. Show that its chromatic number $\chi(G)$ is less than or equal to $d + 1$.

Comment: Too many of you tried to do induction on d. That is not the right strategy here. You need to induct on the number of vertices.

Solution: Suppose $G = (V, E)$ is a graph all of whose vertices are of degree $\leq d$. Let $m = \max_{v \in V} \deg v$. Then $m \leq d$. Let $\mathbf{v}$ be the number of vertices in G. If $\mathbf{v} = 1$, then clearly the statement is true, for in this case $\chi(G) = 1$ and $1 \leq d + 1$, for $d \geq 0$. Let $n > 1$ and suppose the statement is true for all graphs G' such that the number of vertices of G' is $n - 1$ and such that all vertices of G' have degree $\leq d$. Let x be any vertex of G, and G' be the graph obtained be removing x from G as well as all the edges incident on x. Then the number of vertices in G' in $n - 1$ and every vertex of G' has degree $\leq d$. Therefore by induction hypothesis, we can colour G' with $\leq d + 1$ colours. Do so. Since $\deg x \leq d$, x has at most d adjacent vertices. Therefore at most d colours are used up in G' to colour the vertices adjacent to x in G. It is clear that one can assign a colour to x such that G has at most $(d + 1)$ colours. □

(4) (20 marks) Suppose all vertices of a graph have degree 15. Show that the number of edges in G is a multiple of 15.

Solution: Let $\mathbf{e}$ be the number of edges of G, and $\mathbf{v}$ the number of vertices. We know that

$$\sum_{v \in V} \deg v = 2\mathbf{e}$$

where V is the set of vertices of G. Since $\deg v = 15$ for every $v \in V$, the above equation translates to

$$15\mathbf{v} = 2\mathbf{e}.$$

This means 15 divides $2\mathbf{e}$. Now the greatest common divisor of 15 and 2 is 1. By unique factorisation theorem for integers it follows that 15 must divide $\mathbf{e}$. $\square$

(5) (20 marks) Let G be a connected planar graph with eight vertices and 12 edges. Show that a planar depiction of G must have at least three regions of degree ≤ 4.

Solution: As usual, let $\mathbf{e}$ and $\mathbf{v}$ be the number of edges and vertices of G and $\mathbf{r}$ the number of regions determined by a planar depiction of G. Then by Euler's formula we know that $\mathbf{r} - \mathbf{e} + \mathbf{v} = 2$. In our case $\mathbf{v} = 8$ and $\mathbf{e} = 12$, which means $\mathbf{r} = 2 + 12 - 8 = 6$. Thus we have six regions determined by G. Let R_1, R_2, R_3, R_4, R_5, and R_6 be the regions determined by G. We know that

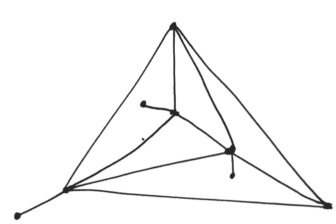
$$\sum_{i=1}^{6} \deg R_i = 2\mathbf{e} = 24.$$

Let n be the number of regions with degree ≤ 4. Then we have $6-n$ regions of degree greater than or equal to 5. Re-order the regions if necessary, and suppose R_1, ..., R_n have degree ≤ 4, and R_{n+1}, R_{n+2}, ..., R_6 have degree ≥ 5. Now all regions have degree at least 3. This means

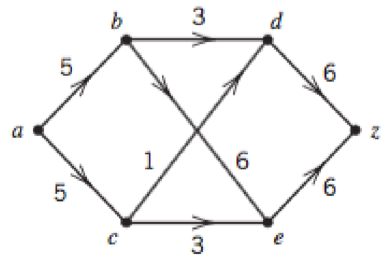
$$\begin{aligned} 24 &= \sum_{i=1}^{6} \deg R_i \\ &= \sum_{i=1}^{n} R_i + \sum_{i=n+1}^{6} R_i \\ &\geq 3n + 5(6-n) \end{aligned}$$

since $\deg R_i \geq 3$ for $1 \leq i \leq n$ and $\deg R_j \geq 5$, $n+1 \leq j \leq 6$. This means $24 \geq 30 - 2n$. In other words $2n \geq 6$, i.e., $n \geq 3$. $\square$

Comment: This finishes the problem. But you might wonder whether it is possible to have three regions of degree ≤ 4 (perhaps the minimum number required is greater). Here is an example of a planar graph with eight vertices and 12 edges with exactly three regions of degree ≤ 4.



(6) Consider the following directed a-z network N with capacities as displayed.



(a) (5 marks) Which of the following is not a flow for N? (The picture for N is reproduced below again for your convenience.)

(i) $f(\vec{a},b) = 5,\ f(\vec{a},c) = 5,\ f(\vec{b},d) = 3,\ f(\vec{b},e) = 2,$
$f(\vec{c},d) = 1,\ f(\vec{c},e) = 4,\ f(\vec{d},z) = 4,\ f(\vec{e},z) = 6.$

(ii) $f(\vec{a},b) = 5,\ f(\vec{a},c) = 3,\ f(\vec{b},d) = 3,\ f(\vec{b},e) = 2,$
$f(\vec{c},d) = 1,\ f(\vec{c},e) = 2,\ f(\vec{d},z) = 4,\ f(\vec{e},z) = 4.$

(iii) $f(\vec{a},b) = 5,\ f(\vec{a},c) = 4,\ f(\vec{b},d) = 3,\ f(\vec{b},e) = 2,$
$f(\vec{c},d) = 1,\ f(\vec{c},e) = 3,\ f(\vec{d},z) = 4,\ f(\vec{e},z) = 5.$

Solution: It is easy to see that for (i), (ii), and (iii), if $v \notin \{a,z\}$ then $\sum_{e\in \mathrm{In}(x)} f(e) = \sum_{e\in \mathrm{Out}(x)} f(e)$. However in (i) we have $f(\vec{c},e) = 4 > k(\vec{c},e) = 3$, which violates the definition of a flow. Thus the function f in (i) is not a flow. It is easy to see that (ii) and (iii) are satisfy the requirement that the flow through an edge is less than or equal to the capacity through that edge. Thus the answer is that (i) is not a flow for N (or more precisely, the f in (i) is not a flow for N), but (ii) and (iii) are flows for N. □

(b) (5 marks) For a directed network, define the cut $(P,\overline{P})$ associated with a set P of vertices of the network.

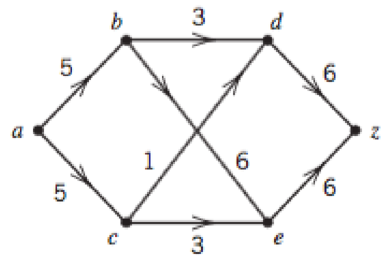
Solution: Let $\overline{P}$ denote the set of vertices in the directed network which are not in P. The cut $(P,\overline{P})$ associated with P is the set of all edges $(\vec{x},y)$ in the directed network with $x \in P$ and $y \in \overline{P}$.

Comment: Too many of you tried to define a cut for the network N shown in the picture. You were asked for a general definition.

(c) (10 marks) Find a maximum flow for the network N displayed on the previous page (and below) and show it is a maximum flow by finding a minimum a-z cut $(P,\overline{P})$ for that flow. Prove that your cut is indeed the minimum cut for your flow.

Solution to (c) on next page.

Solution: Here is N.



Consider the set $P = \{a,\, c\}$. Then $(P,\, \overline{P}) = \{(\vec{a}, b), (\vec{c}, d), (\vec{c}, e)\}$. Note that $(\overline{P}, P)$ is empty. The capacity of the cut $(P, \overline{P})$ is

$$k(P, \overline{P}) = k(\vec{a}, b) + k(\vec{c}, d) + k(\vec{c}, e) = 5 + 1 + 3 = 9.$$

If we can find a flow f for N such that $|f| = k(P, \overline{P})$ then we know that f is a maximum flow and $(P, \overline{P})$ a minimum a-z cut for that flow (see Corollary 2(a) on page 138 of the text, especially the last line). Now consider the flow in (iii) above. We have

$$\begin{aligned} |f| &= \sum_{e \in \mathrm{Out}(a)} f(e) \\ &= f(\vec{a}, b) + f(\vec{a}, c) \\ &= 5 + 4 \\ &= 9 \\ &= k(P, \overline{P}). \end{aligned}$$

Thus the flow in (iii) is a maximum flow, and $(P, \overline{P})$ is a minimum a-z cut for (iii).

Comment: One way of figuring out the problem is this. The sum of the capacities of the edges going out from a is 10, but no flow f can possibly have $|f| = 10$, because such a flow would have to have $f(\vec{a}, c) = 5$ and this is greater than the sum of the capacities of edges going out of c. On the other hand if we look at the flow in (iii), it has the next best number as its value, namely 9, and so it has to be a maximum flow. Finding a minimum cut for (iii) is easy after that.