QUIZ 6
OCTOBER 10, 2018

Denote by m the Lebesgue measure on $\mathbb{R}$ divided by $\sqrt{2\pi}$.

$$m = \frac{1}{\sqrt{2\pi}}(\text{Lebesgue measure}).$$

Let

$$H(t) = e^{-|t|},$$

and for $\lambda > 0$ set

$$H_\lambda(t) = H(\lambda t),$$

and

$$h_\lambda(x) = \int_{-\infty}^{\infty} H_\lambda(t) e^{itx}\, dm(t) = \sqrt{\frac{2}{\pi}} \frac{\lambda}{\lambda^2 + x^2}.$$

(1) Let $f \in L^1$ and $\widehat{f}$ its Fourier Transform. Prove that

$$(f * h_\lambda)(x) = \int_{-\infty}^{\infty} H_\lambda(t) \widehat{f}(t) e^{ixt}\, dm(t).$$

(2) If $f \in L^2$ and $\widehat{f} \in L^1$, show that

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(t) e^{ixt}\, dm(t).$$

Here $\widehat{f}$ is Fourier transform on L^2 obtained by extending the usual Fourier transform from $L^2 \cap L^1$. [Hint: Consider ψ_A, the inverse Fourier transform of $\widehat{f}\chi_{[-A,A]}$ for $A > 0$, and use a theorem proved in class on Saturday.]