Oct 24, 2018

Quiz 5

1. Let $\{n_k\}$ be a strictly increasing sequence of positive integers. Let $E = \{x \in [0,2\pi) \mid \{\sin(n_k x)\}_{k=1}^{\infty} \text{ converges}\}$. Show that $m(E)=0$, where m is the Lebesgue measure on $[0,2\pi)$. [Hint: You may use the fact that $\lim_{n\to\infty}\int_A \sin nx\,dx = \lim_{n\to\infty}\int_A \cos nx\,dx = 0$ for every non-empty m'ble set A in $[0,2\pi)$.]

2. Let $(X,\mathcal{S},\mu)$ and $(Y,\mathcal{T},\nu)$ be σ-finite measure spaces. For $Q \subseteq X\times Y$, $x\in X$, and $y \in Y$ define

$$Q_x = \{y\in Y \mid (x,y)\in Q\}$$
$$Q^y = \{x\in X \mid (x,y)\in Q\}.$$

Remark: Q_x is "essentially" the intersection of Q with the fibre over x of $X\times Y \longrightarrow X$, and a similar description works for Q^y.

Assume:

(i) $E\in \mathcal{S}\times\mathcal{T} \Rightarrow E_x \in \mathcal{T}$ $\forall x\in X$ and $E^y \in \mathcal{S}$ $\forall y \in Y$.

(ii) For $E\in\mathcal{S}\times\mathcal{T}$ if $\varphi_E : X\longrightarrow[0,\infty]$ and $\psi_E : Y\longrightarrow[0,\infty]$ are the maps $x\mapsto \nu(E_x)$ and $y\mapsto\mu(E^y)$ respectively then φ_E and ψ_E are measurable.

With the above assumptions, for $E\in\mathcal{S}\times\mathcal{T}$, define $\sigma(E)=\int_X \varphi_E\,d\mu$ and $\bar{\sigma}(E) = \int_Y \psi_E\,d\nu$.

Show

[Hint: You may need MCT.]

(a) σ is a measure on $(X\times Y, \mathcal{S}\times\mathcal{T})$. (By symmetry so is $\bar{\sigma}$.)

(b) $\sigma = \bar{\sigma}$ on $\mathcal{S}\times\mathcal{T}$. [Hint: Test the assertion on measurable rectangles. You may use results from HW problems provided you quote the results accurately.]