Sep 12, 2018 Quiz 3

Throughout $(X, \mathcal{M})$ is a measurable space. All measures are on $\mathcal{M}$.

Definitions: Let μ and ν be measures.

(a) ν is said to be absolutely continuous with respect to μ, if for every $E \in \mathcal{M}$ s.t. $\mu(E) = 0$, we have $\nu(E) = 0$.

(b) μ is said to be concentrated in a set C in $\mathcal{M}$ if $\mu(E) = \mu(E \cap C)$ for every $E \in \mathcal{M}$. (Equivalently $\mu(E) = 0$ for every E disjoint from C).

(c) μ and ν are mutually singular, written $\mu \perp \nu$, if there exist disjoint sets A and B in $\mathcal{M}$ with μ concentrated on A and ν concentrated on B.

Examples: If $f \geq 0$ is measurable then $\nu(E) := \int_E f \, d\mu$ defines a measure ν such that $\nu \ll \mu$. On the other hand, if $x \in \mathbb{R}$, then the Dirac measure on $\mathbb{R}$, δ_x, and the Lebesgue measure m on $\mathbb{R}$ are mutually singular.

Problems and solutions start on the next page

Problems:

1. Let ν, μ, λ be measures. Show that

(a) $\nu \perp \lambda$ and $\mu \perp \lambda \Rightarrow \nu + \mu \perp \lambda$.

(b) $\nu \perp \lambda$ and $\mu << \lambda \Rightarrow \nu \perp \mu$.

(c) $\nu \perp \mu$ and $\nu << \mu \Rightarrow \nu = 0$.

Solution

(a) Suppose A_ν (resp. A_μ) and B_ν (resp. B_μ) are disjoint measurable sets such that ν (resp. μ) is concentrated in A_ν (resp. A_μ) and λ is concentrated in B_ν as well as in B_μ. Let $A = A_\mu \cup A_\nu$ and $B = B_\nu \cap B_\mu$. Clearly $A \cap B = \phi$, $\mu + \nu$ is concentrated in A and λ in B. Indeed if $C \cap A = \phi$, then $C \cap A_\mu = C \cap A_\nu = \phi$, and hence $\mu(C) = \nu(C) = 0$, for $C \in \mathfrak{M}$. Hence $\mu + \nu$ is concentrated in A.

If $E \in \mathfrak{M}$, then $E \cap B = (E \cap B_\mu) \cap B_\nu$, whence $\lambda(E \cap B) = \lambda(E \cap B_\mu) = \lambda(E)$. Thus λ is concentrated in B. Since A and B are disjoint, $\mu + \nu \perp \lambda$.

(b) Let A and B be disjoint measurable sets such that ν is concentrated in A and λ in B, and $A \cap B = \phi$. Since $\mu << \lambda$, by definition μ is concentrated in B, for $\mu(C) = 0$ for every $C \in \mathfrak{M}$ disjoint from B (indeed $\lambda(C) = 0$). Thus $\mu \perp \nu$.

(c) Suppose A, B are disjoint m'ble sets with ν concentrated in A and μ in B. Such (A, B) exist since $\nu \perp \mu$. Since $\mu(A) = 0$ and $\nu << \mu$, it follows that $\nu(A) = 0$. Thus $\nu(C) = \nu(C \cap A) = 0$ $\forall$ $C \in \mathfrak{M}$, showing $\nu = 0$. //

2. Show that if a measurable function f is such that $\int_E f\,d\mu = 0$ for every $E \in \mathcal{M}$, then $f = 0$ a.e. $[\mu]$.

Solution:

Since $\int_E f\,d\mu$ exists $\forall\, E \in \mathcal{M}$, $f \in L^1(\mu)$. Breaking up f into its real and imaginary parts, we may assume f is real valued. For $n \in \mathbb{N}$, let $E_n = \{f > \frac{1}{n}\}$. Then

$$0 = \int_{E_n} f\,d\mu \geq \int_{E_n} \tfrac{1}{n}\,d\mu = \tfrac{1}{n}\,\mu(E_n) \geq 0.$$

Thus $\mu(E_n) = 0$ $\forall\, n$. Since $\{f > 0\} = \bigcup_{n=1}^{\infty} E_n$, it follows that $\mu(\{f > 0\}) = 0$. Similarly $\mu(\{f < 0\}) = 0$. Thus $\mu(\{f \neq 0\}) = 0$. The assertion follows. //

3. Suppose μ and ν are finite measures with $\nu \leq \mu$ (i.e., $\nu(E) \leq \mu(E)$ $\forall\, E \in \mathcal{M}$). Suppose g is a real measurable function such that $\int_E g^+\,d\mu$ and $\int_E g^-\,d\mu$ are never simultaneously ∞ for any $E \in \mathcal{M}$, and such that for every $E \in \mathcal{M}$, we have

$$(*) \qquad \nu(E) = \int_E g\,d\mu$$

(a) Show that $0 \leq g \leq 1$ a.e. $[\mu]$, and hence a.e. $[\nu]$.

(b) From (a), WLOG we may assume that $0 \leq g(x) \leq 1$ for all $x \in X$. Let $A = \{x \in X \mid g(x) > 0\}$. Show that

$$\mu(E \cap A) = \int_E \frac{1}{g}\,d\nu \qquad \forall\, E \in \mathcal{M}.$$

Solution

(a) $\nu(\{g = \infty\}) = \int_{\{g=\infty\}} g\,d\mu = \infty \cdot \mu\{g = \infty\}$. Since the left side is finite, this means $\mu\{g = \infty\} = 0$.

Next, for $n \in \mathbb{N}$, let $E_n = \{g \geq 1 + \frac{1}{n}\}$, and

$E = \{ g > 1 \}$. Then $E_n \subset E_{n+1}$ $\forall n \in \mathbb{N}$, and $\bigcup_n E_n = E$.

Thus $\mu(E) = \lim_{n \to \infty} \mu(E_n)$.

Now

$$\nu(E_n) = \int_{E_n} g \, d\mu \geqslant (1+\tfrac{1}{n}) \int_{E_n} d\mu$$

$$= (1+\tfrac{1}{n}) \, \mu(E_n)$$

$$\geqslant (1+\tfrac{1}{n}) \, \nu(E_n) \quad (\text{for } \nu \leq \mu).$$

It follows that all members in the chain of inequalities above are zero, since $\nu(E_n) < \infty$. Thus $\mu(E_n) = 0$, whence $\mu(E) = 0$.

Similarly,

$$0 \leq \nu(\{ g < -\tfrac{1}{n} \}) = \int_{\{g < -\frac{1}{n}\}} g \, d\mu$$

$$\leq -\tfrac{1}{n} \, \mu(\{ g < -\tfrac{1}{n} \})$$

$$\leq 0$$

Thus all terms in the above chain of equalities and inequalities are zero. In particular $\mu(\{ g < -\frac{1}{n} \}) = 0$. It follows that $\mu(\{ g < 0 \}) = 0$. Thus $0 \leq g \leq 1$ a.e. $[\mu]$. //

(b) We have

$$\nu(A^c) = \int_{A^c} g \, d\mu = \int_{\{g=0\}} g \, d\mu = 0.$$

Let $E \in \mathcal{M}$. Since $g > 0$ on $E \cap A$, we have, from a HW exercise (see problem 9(b) of HW 5)

$$\mu(E\cap A) = \int_{E\cap A} \frac{1}{g}\, d\nu \qquad \text{(from HW exercise)}$$

$$= \int_{E\cap A} \frac{1}{g}\, d\nu + \int_{E\cap A^c} \frac{1}{g}\, d\nu \qquad (\text{since } \nu(A^c)=0)$$

$$= \int_E \frac{1}{g}\, d\nu. \quad //$$

<u>Remark</u>: The above can also be proven as follows. Consider $\phi = 1-g$. Then $0 \le \phi \le 1$. Also $1-\phi^{n+1} = (1-\phi)(1+\phi+\phi^2+\dots+\phi^n)$. On A^c, $\phi \equiv 1$, whence $1-\phi^{n+1} \equiv 0$ on A^c. On A, $0 \le \phi < 1$, and hence $1-\phi^{n+1}$ increases to χ_A as $n \to \infty$. On the other hand $(1-\phi)(1+\phi+\dots+\phi^n) = g\,(1+\phi+\dots+\phi^n)$. Thus

$$\int_E (1-\phi^{n+1})\, d\mu = \int_E g\,(1+\phi+\dots+\phi^n)\, d\mu = \int_E (1+\phi+\dots+\phi^n)\, d\nu.$$

The last equality follows from the Theorem on page 3 of Lecture 4b (i.e., from page 3 of the second part of Lecture 4).

Now $1+\phi+\phi^2+\dots+\phi^n$ increases to $\frac{1}{1-\phi} = \frac{1}{g}$, with the understanding that in this case both sides are ∞ on A^c.

Thus $\int_E (1-\phi^{n+1})\, d\mu \xrightarrow{\text{MCT}} \int_E \chi_A\, d\mu = \mu(A\cap E)$ as $n \to \infty$

On the other hand

$$\int_E (1+\phi+\dots+\phi^n)\, d\nu \xrightarrow{\text{MCT}} \int_E \frac{1}{g}\, d\nu \quad \text{as } n \to \infty.$$

Thus $\mu(A\cap E) = \int_E \frac{1}{g}\, d\nu.$

However, the simplest way of seeing this is the following:-

Proposition: Suppose $h : X \longrightarrow [0,\infty]$ is measurable. Let $B = \{x \in X \mid 0 < h(x) < \infty\}$. Then

$$\frac{1}{h} \cdot h = \chi_B .$$

The proof is left as an exercise for you. It is a simple exercise of the conventions governing products involving ∞ and 0. Note that the proof of the Theorem on p.3 of Lecture 4b respects these conventions. Now consider our g.

We have $0 \le g \le 1$. Hence $\{g = \infty\} = \phi$, i.e., $\{g > 0\} = \{0 < g < \infty\}$, i.e., $A = \{0 < g < \infty\}$. From the Proposition above, we get $\frac{1}{g} \cdot g = \chi_A$.

From here, proving the result is easy. Indeed,

$$\int_E \frac{1}{g}\, d\nu = \int_E \frac{1}{g}\, g\, d\mu = \int_E \chi_A \, d\mu = \mu(E \cap A).$$

Using the result on p.3, Lecture 4b.