Sep 12, 2018 Quiz 3

Throughout $(X, \mathcal{M})$ is a measurable space. All measures are on $\mathcal{M}$.

Definitions: Let μ and ν be measures.

(a) ν is said to be absolutely continuous with respect to μ, if for every $E \in \mathcal{M}$ s.t. $\mu(E) = 0$, we have $\nu(E) = 0$.

(b) μ is said to be concentrated in a set C in $\mathcal{M}$ if $\mu(E) = \mu(E \cap C)$ for every $E \in \mathcal{M}$. (Equivalently $\mu(E) = 0$ for every E disjoint from C).

(c) μ and ν are mutually singular, written $\mu \perp \nu$, if there exist disjoint sets A and B in $\mathcal{M}$ with μ concentrated on A and ν concentrated on B.

Examples: If $f \geq 0$ is measurable then $\nu(E) := \int_E f \, d\mu$ defines a measure ν such that $\nu \ll \mu$. On the other hand, if $x \in \mathbb{R}$, then the Dirac measure on $\mathbb{R}$, δ_x, and the Lebesgue measure m on $\mathbb{R}$ are mutually singular.

Problems:

1. Let ν, μ, λ be measures. Show that

(a) $\nu \perp \lambda$ and $\mu \perp \lambda \Rightarrow \nu + \mu \perp \lambda$.

(b) $\nu \perp \lambda$ and $\mu \ll \lambda \Rightarrow \nu \perp \mu$.

(c) $\nu \perp \mu$ and $\nu \ll \mu \Rightarrow \nu = 0$.

2. Show that if a measurable function f is such that $\int_E f\,d\mu = 0$ for every $E \in \mathcal{M}$, then $f = 0$ a.e. $[\mu]$.

3. Suppose μ and ν are finite measures with $\nu \leq \mu$ (i.e., $\nu(E) \leq \mu(E)$ $\forall E \in \mathcal{M}$). Suppose g is a real measurable function such that $\int_E g^+ \,d\mu$ and $\int_E g^- \,d\mu$ are never simultaneously ∞ for any $E \in \mathcal{M}$, and such that for every $E \in \mathcal{M}$, we have

(*) $$\nu(E) = \int_E g\,d\mu$$

(a) Show that $0 \leq g \leq 1$ a.e. $[\mu]$, and hence a.e. $[\nu]$.

(b) From (a), WLOG we may assume that $0 \leq g(x) \leq 1$ for all $x \in X$. Let $A = \{x \in X \mid g(x) > 0\}$. Show that

$$\mu(E \cap A) = \int_E \frac{1}{g}\,d\nu \qquad \forall E \in \mathcal{M}.$$

Remark: Recall that modulo some facts about Hilbert spaces and $L^2(\mu)$, we showed in HW 5 that if μ is finite, and $\nu \leq \mu$, then $\exists\, g \in L^1(\mu)$ s.t. (*) holds for every $E \in \mathcal{M}$. By problem 2 above, this g is unique a.e. $[\mu]$. From these two facts it is not hard to see that g satisfying (*) $\forall E \in \mathcal{M}$ exists even when μ is σ-finite and $\nu \leq \mu$. To see this, break up X into disjoint pieces on each of which μ is finite and use g from each piece.