## QUIZ 2

**Riemann Integrals.** In what follows $I = [0,1]$ and $C(I)$ is the space of continuous functions on $I$. The norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are the ones defined earlier in your homework problems. You may use Cauchy-Schwarz.

(1) Let $\{k_n\}$ be a sequence of continuous real-valued functions on $I$ such that $k_n(t) \geq 0$ for all $n \in \mathbb{N}$ and all $t \in I$. Suppose $p$ is a continuous function on $I$ such that $p \geq 0$ on $I$ and $p(t) \leq \sum_{n=1}^{\infty} k_n(t)$ for every $t \in I$. Show that

$$\int_0^1 p(t)dt \leq \sum_{n=1}^{\infty} \int_0^1 k_n(t)dt.$$

(Warning: $\sum_{n=1}^{\infty} k_n(t)$ could diverge for some (even all) $t \in I$.)

(2) Let $\{h_n\}$ be a sequence in $C(I)$ such that $0 \leq h_n \leq 1$ for all $n \in \mathbb{N}$, $h_n \to 0$ pointwise on $I$ as $n \to \infty$, and $\int_0^1 h_n(t)dt \to L$ as $n \to \infty$. Show that if $\sum_{n=1}^{\infty} \|h_n - h_{n+1}\|_2 < \infty$ then $h_n = \sum_{m=n}^{\infty}(h_m - h_{m+1})$.

(3) Let $\{h_n\}$, $L$ be as in Problem (2). Show that $L = 0$

**Inner Product Spaces.** Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$ and $V$ a vector space over $\mathbb{K}$. Recall that an *inner product on* $V$ is a map

$$\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{K}$$

such that for $x$, $y$, and $z$ in $V$ and $\alpha \in \mathbb{K}$ the following conditions are satisfied. (a) $\langle x, x\rangle \geq 0$, (b) $\langle x, x \rangle = 0 \Leftrightarrow x = 0$, (c) $\langle x, y\rangle = \overline{\langle y, x\rangle}$, (d) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$, and $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$. An *inner product space* over $\mathbb{K}$ is a vector space over $\mathbb{K}$ with an inner product. If $(V, \langle\cdot,\cdot\rangle)$ is an inner product space, the map

$$\|\cdot\| \colon V \to \mathbb{R}$$

given by

$$\|x\| = \sqrt{\langle x, x\rangle} \qquad (x \in V)$$

is a norm on $V$.

(4) (*Polarization Identity*) Show that in any inner product space over $\mathbb{C}$

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right).$$

(Once again, you may use the Cauchy-Schwarz inequality for inner product spaces, namely $|\langle x, y\rangle| \leq \|x\|\|y\|$ for all $x$ and $y$ in $V$.)