Topics Covered:

1. Baire category theorem
2. Uniform boundedness principle
3. Open-mapping Theorem
4. Closed-Graph Theorem.

1. Baire category Theorem

let X be a complete metric space. For each countable collection of open dense sets $\{U_n\}_{n\geq 1}$, their intersection is also dense

Equivalently

X is not a countable union of nowhere dense sets. ie, if $X = \bigcup_{n\geq 1} F_n$, F_n are closed sets then $\exists$ $n_0 \in \mathbb{N}$ s.t $(F_{n_0})^{\circ} \neq \phi$.

Proof:

if $\{U_n\}_{n\geq 1}$ is a countable collection of open dense sets. we want to show $\bigcap_{n\geq 1} U_n$ is dense.

It is sufficient to show that for any non-empty open set W in X, $\bigcap_{n\geq 1} U_n \cap X \neq \phi$.

Since U_1 is dense, $W \cap U_1 \neq \phi$.

$\exists$ x_1 & $0 < r_1 < 1$ s.t

$\bar{B}(x_1, r_1) \subset W \cap U_1$

where $B(x_1, r_1)$ is open ball of radius r_1 around the point x_1,
similarly $\bar{B}(x_1, r_1)$ is a closed ball.

we can recursively find a pair x_n & $0 < r_n < \frac{1}{n}$ s.t

$\bar{B}(x_n, r_n) \subseteq B(x_{n-1}, r_{n-1}) \cap U_n$

Since $x_n \in B(x_m, r_m)$ if $n > m$, x_n is cauchy

so $x_n \longrightarrow x$ for some $x \in X$.

By closedness $x \in \overline{B}(x_n, r_n)$

$\Rightarrow x \in W$ & $x \in U_n \ \forall n$.

$\blacksquare$

2. Uniform Boundedness Principle

let X, Y Banach spaces. Suppose F is a collection of continuous linear operators from X to Y. If $\forall x \in X$

$$\sup_{T \in F} \|T(x)\|_Y < \infty \quad \text{(call it } M\text{)}$$

Then $\sup_{T \in F} \|T\|_{B(X,Y)} < \infty$.

Proof :

$\forall n \in \mathbb{N}$.

$$X_n = \{ x \in X : \sup_{T \in F} \|T(x)\|_Y \leq n \}$$

X_n is a closed set and by assumption

$$\bigcup_{n \in N} X_n = X$$

By Baire category theorem $\exists\, m$ s.t X_m has non-empty interior

By Baire category theorem $\exists\, m$ s.t $\overline{B_\epsilon(x_0)} \subseteq X_m$.

ie, $\exists\, x_0 \in X_m$ & $\epsilon > 0$ s.t $B_\epsilon(x_0) \subseteq X_m$.

let $u \in X$ with $\|u\| \leq 1$ & $T \in F$

$$\|T(u)\|_Y = \frac{1}{\epsilon} \|T(x_0 + \epsilon u) - T(x_0)\|_Y$$

$$\leq \frac{1}{\epsilon} \|T(x_0 + \epsilon u)\|_Y + \|T(x_0)\|_Y$$

$$\leq \frac{1}{\epsilon}(m + m) \quad \text{because } x_0 + \epsilon u \text{ \& } x_0 \in X_m$$

Thus $\sup_{T \in F} \|T\|_{B(X,Y)} \leq \frac{2m}{\epsilon} < \infty$ $\blacksquare$

3. OPEN MAPPING THEOREM :

Let X, Y be Banach spaces, $A : X \to Y$ be surjective bounded linear operator. Then A is a open map.

Proof :

Let $U = B_X(0,1)$, $V = B_Y(0,1)$

Then $X = \bigcup_{k \in \mathbb{N}} kU$

Since A is surjective

$$Y = A(X) = \bigcup_{k \in \mathbb{N}} A(kU) = \bigcup_{k \in \mathbb{N}} \overline{A(kU)}.$$

Since Y is Banach, By Baire category Theorem

$\exists\, k \in \mathbb{N}$, $\overline{A(kU)}^{\circ} \neq \phi$.

$\exists\, r > 0$ & c in Y s.t

$$\overline{B(c,r)} \subseteq \overline{A(kU)}^{\circ} \subseteq \overline{A(kU)}$$

Let $v \in V = B_Y(0,1)$

then $c, c+rv \in \overline{A(kU)}$

So $rv \in \overline{A(kU)} + \overline{A(kU)} \subseteq \overline{A(kU) + A(kU)} \subseteq \overline{A(2kU)}$.

Due to the continuity of $+$.

Sothat

$$v \in \overline{A\left(\frac{2k}{r}U\right)} \qquad \text{call } \frac{2k}{r} = L.$$

$$v \in \overline{A(LU)}$$

it follows that $\forall\, y \in Y$, $\forall\, \epsilon > 0$ $\exists\, x \in X$;

$$\|x\|_X \leq L\|y\|_Y \;\; \& \;\; \|y - Ax\|_Y < \epsilon \qquad \text{——— (1)}$$

claim $V \subseteq A(2LU)$

Let $y \in V$ by (1) $\exists\, x_1$ with $\|x_1\| < L$ & $\|y - Ax_1\| < \frac{1}{2}$

now inductively define x_n s.t.

$$\|x_n\| < \frac{L}{2^{n-1}} \;\; \& \;\; \|y - A(x_1 + \dots x_n)\| < \frac{1}{2^n} \qquad \text{——— (2)}$$

call $S_n = x_1 + x_2 \dots x_n$

From (2)

S_n is cauchy, so $S_n \longrightarrow x$ for some $x \in X$

From (2) $AS_n \longrightarrow y$.

So by continuity $A(x) = y$

$\|x\| \leq \sum_{n=1}^{\infty} \|x_n\| < 2L$.

$\Rightarrow$ $y \in A(2LU)$ Thus $V \subseteq A(2LU)$

So $A(U)$ image of the open ball contains a open ball $\frac{V}{2L}$ of Y.

Thus $A(U)$ is a neighbourhood of 0 in Y ■

4. Closed Graph Theorem :

Let $T: X \longrightarrow Y$ be a linear operator

T is continuous iff $G(T)$ (Graph of T) is closed in $X \times Y$.

$\Rightarrow$ if T is continuous clearly Graph is closed, its the kernal of

$$A: X \times Y \longrightarrow Y$$
$$(x, y) \longmapsto y - T(x)$$

$\Leftarrow$

$$\begin{tikzcd} & G(T) \subseteq X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

Since $X \times Y$ is banach, $G(T)$ is closed in $X \times Y$, $G(T)$ is banach.

π_1 is surjective, continuous map so it is a open map $\pi_1 : G(T) \longrightarrow X$ is a bijection

So the inverse map $B: X \longrightarrow G(T)$ is bounded.
$$x \longmapsto (x, T(x))$$

so $T = \pi_2 \circ B$ so it is bounded.