Tutorial

Theorem:

Let $A \subseteq \mathbb{R}^1$, every subset of A is lebesgue measurable then $m(A) = 0$.

※ Here m & μ represent the same lebesgue measure on $\mathbb{R}^1$

Proof:

Now consider the group $(\mathbb{R}, +)$
make a equivalence relation of $\mathbb{R}$
$x \sim y$ if $x - y \in \mathbb{Q}$. (check this is an eq. relation)
Choose a representative from each equivalence class and form a set "E".
(The above claim uses Axiom of choice)

E has the following properties

(i) $(E+r) \cap (E+s) = \phi$ if $r, s \in \mathbb{Q}$ & $r \neq s$.

(ii) $\bigsqcup_{r \in \mathbb{Q}} E + r = \mathbb{R}$

Pf:

(i) Suppose $(E+r) \cap (E+s) \neq \phi$

$x \in (E+r) \cap (E+s)$ $\qquad e, e' \in E$

$x = e + r = e' + s$

$\Rightarrow e - e' = -r + s \in \mathbb{Q}$

ie $e \sim e'$ (contradicts the construction of E)

(ii) clearly.

for now fix $t \in \mathbb{Q}$, Define $A_t = A \cap (E+t)$.

⟿ By hypothesis A_t is measurable

claim $\mu(A_t) = 0$

Enough to show $\mu(K) = 0$ $\forall$ $K \subseteq A_t$; K compact
(By regularity)

Let H be the union of translates of K by r, $r \in \mathbb{Q} \cap [0,1]$

H is bounded, $\mu(H) < \infty$

Since $K \subset E + t$, $K + r$ are pairwise disjoint.

So $m(H) = \sum_r m(K + r)$

use translation invariance $\mu(K+r) = \mu(K)$.

$\Rightarrow \mu(K) = 0$.

$\Rightarrow \mu(A_t) = 0$

$A = \bigcup_{t \in \mathbb{Q}} A_t \Rightarrow \mu(A) = 0$

Corollary:

Every set with positive measure has a non-measurable subset.

Lusin's Theorem:

Suppose f is complex measurable on X.
$f(x) = 0$ if $x \notin A$, $\mu(A) < \infty$.

$\forall\ \epsilon > 0$, there exist a $g \in C_c(X)$ s.t

$$\mu(\{x : f(x) \neq g(x)\}) < \epsilon$$

Furthermore,

$$\sup_{x \in X} |g(x)| \leq \sup_{x \in X} |f(x)|$$

Proof:

Case 1: Assume $0 \leq f < 1$ & K is compact.

$\exists$ sequence of simple functions.

$$s_1 \leq s_2 \ldots \ldots s_n \leq \ldots \uparrow f \quad (\text{pt. wise})$$

put $t_1 = s_1$, $t_2 = s_2 - s_1$, ... $t_n = t_n - t_{n-1}$

$2^i t_i$'s are simple characteristic functions (check) on $T_n \subseteq A_0 = A$

$$f(x) = \sum_{n=1}^{\infty} t_n(x) \quad (x \in X)$$

Fix a open set V s.t $A \subset V$ and $\bar{V}$ is compact.
(possible because A is compact & X is locally compact)

$\exists$ compact set K_n and open set V_n s.t
$K_n \subseteq T_n \subseteq V_n \subseteq V$ and $\mu(V_n \setminus K_n) < 2^{-n}\epsilon$

By urysohn's lemma $\exists$ h_n s.t $K_n \prec h_n \prec V_n$.

Define $$g(x) = \sum_{n=1}^{\infty} \frac{h_n(x)}{2^n}$$

The convergence is uniform, so g is continuous.
support of $g \subseteq \bar{V}$; $\Rightarrow g \in C_c(X)$.

$2^{-n} h_n$ & t_n differ only in the set $V_n \setminus K_n$

so g & h differ in a subset of $\cup (V_n \setminus K_n)$

whose measure is less than ϵ.

General case:
Similar thing can be done for any bounded measurable functions & A compact.

The compactness of A can be removed.
if A is not-compact, find K, compact
s.t $\mu(K \setminus A) < \epsilon$ for any given ϵ.

If f is a complex measurable function
take $B_n = \{x \mid f(x) > n\}$ $\quad \cap B_n = \phi$.

$B_1 \supseteq B_2 \supseteq \ldots\ldots$

$\mu(B_n) \longrightarrow 0$

f coincides with bdd measurable function $(1 - \chi_{B_n}) \cdot f$
except on B_n.

The supremum can be adjusted in the following way.

$R = \sup \{ |f(x)| : x \in X \}$.

define $\phi(z) = \begin{cases} z & \text{if } |z| \le R \\ \frac{Rz}{|z|} & \text{if } |z| > R \end{cases}$

This is a continuous function by pasting lemma.

find g as before
& define $g_1 = \phi \circ g$.
g_1 satisfies every condition

<u>Corollary</u>: Assume hypothesis of Lusin's Theorem.
$|f| \le 1$

Then $\exists$ sequence $\{g_n\}$ s.t $g_n \in C_c(X)$, $|g_n| \le 1$ &

$f(x) = \lim_{n \to \infty} g_n(x)$ a.e.

Lusins Theorem can be used to prove $C_c(X)$ is dense in $L^p(X)$. $1 \le p < \infty$