* Every Hilbert space admits a projection operator.

Lemma 1:

* Given: H - Hilbert space.

  C - closed, convex subset of H.

  Let $C \neq H$.

To show: ~~Let~~ ~~$x_0 \in$~~ Let $x_0 \in H \setminus C$.

Then there exist a unique $y_0 \in C$ s.t

$$d(x_0, C) = \|y_0 - x_0\|.$$

Proof: Let $d = d(x_0, C) = \inf\{\|x_0 - y\| : y \in C\}$

Then $d > 0$ as C is closed set.

Then for each $n \in \mathbb{N}$ $\exists$ $y_n \in C$ s.t

$$d \leq \|x_0 - y_n\| < d + 1/n$$

so, $\|x_0 - y_n\| \to d$.

Claim: $\{y_n\}$ is a cauchy sequence.

Proof:
$$\|y_n - y_m\|^2 = \|(x_0 - y_m) - (x_0 - y_n)\|^2$$

$\|x+y\|^2 + \|x-y\|^2 = 2\{\|x\|^2 + \|y\|^2\}$

$$= 2\{\|x_0 - y_m\|^2 + \|x_0 - y_n\|^2\} - \|2x_0 - y_m - y_n\|^2$$

$-d \leq \|x_0 - \frac{y_m + y_n}{2}\|^2$

$$= 2\{\|x_0 - y_m\|^2 + \|x_0 - y_n\|^2\} - 4\|x_0 - \frac{y_m + y_n}{2}\|^2$$

$$\leq 2\|x_0 - y_m\|^2 + 2\|x_0 - y_n\|^2 - 4d^2$$

$\to 0$ as $m, n \to \infty$.

So, $\{y_n\}$ is cauchy sequence in C.

So, $\exists\, y_0 \in C$ s.t $y_n \to y_0$.

Now $d = \lim_{n\to\infty} \|x_0 - y_n\|$

$= \|x_0 - y_0\|$ (by continuity of $\|\cdot\|$)

Now we prove that y_0 is unique.

suppose $z_0 \in C$ such that

$d = \|x_0 - z_0\|$.

Now $\|y_0 - z_0\|^2 = \|(x_0 - z_0) - (x_0 - y_0)\|^2$

$= 2(\|x_0 - z_0\|^2 + \|x_0 - y_0\|^2) - \|2x_0 - (y_0 + z_0)\|^2$

$\leq 2d^2 + 2d^2 - 4d^2 = 0.$

$\Rightarrow \|y_0 - z_0\| = 0 \Rightarrow y_0 = z_0$.

So, y_0 is unique.

Lemma 2: Given: $\mathcal{H}$ - Hilbert space.

M - closed subspace of H.

$x_0 \in H \setminus M$

$y_0 \in M$.

~~Then~~ To show:

$d(x_0, M) = \|x_0 - y_0\| \Leftrightarrow x_0 - y_0 \perp M$.

Proof: Let $x_0 - y_0 \perp M$.

Now let $y \in M$.

$\|x_0 - y\|^2 = \|x_0 - y_0 + y_0 - y\|^2$

$= \|x_0 - y_0\|^2 + \|y_0 - y\|^2 \geq \|x_0 - y_0\|^2$

$\Rightarrow d(x_0, M) = \|x_0 - y_0\|$ (Proved).

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Conversely let,

$$d(x_0, M) = \|x_0 - y_0\|.$$

<u>To show</u>: $x_0 - y_0 \perp M$.

<u>Proof</u>: If possible let $x_0 - y_0 \not\perp M$ i.e

$\exists \; z_0 \in M$ s.t $\langle x_0 - y_0, z_0 \rangle \neq 0$.

Now, $\Big\| x_0 - \underbrace{\Big(y_0 + \frac{\langle x_0 - y_0, z_0 \rangle z_0}{\|z_0\|^2} \Big)}_{M} \Big\|^2$

Take $p = x_0 - y_0$

$$= \Big\| p - \frac{\langle p, z_0 \rangle z_0}{\|z_0\|^2} \Big\|^2$$

$$= \Big\langle p - \frac{\langle p, z_0 \rangle z_0}{\|z_0\|^2}, p - \frac{\langle p, z_0 \rangle z_0}{\|z_0\|^2} \Big\rangle$$

$=$

$$= \|p\|^2 - \frac{|\langle p, z_0 \rangle|^2}{\|z_0\|^2} < \|p\|^2 = \|x_0 - y_0\|^2$$

which is a contradiction.

Conclusion of this lemma is

if $H \neq M$ then $\exists$ non zero element in $M^\perp$ (i.e $x_0 - y_0$).

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Lemma 3: Given: For any closed subspace M.
To show: $H = M \oplus M^{\perp}$ &

Proof: We have $M \cap M^{\perp} = \{0\}$.

let $z \in M \cap M^{\perp}$

$z \in M$ & $z \in M^{\perp} \Rightarrow \langle z, z \rangle = 0$
$\|z\|^2 = 0$
$z = 0$.

Now we prove that
$H = M + M^{\perp}$.
In general sum of two closed subspace need not be closed but for two perpendicular subspace sum of two closed space is closed.

So, $M + M^{\perp}$ is closed.
If possible let $H \neq M + M^{\perp}$.
Then $\exists\, p \in H$, & $p \perp (M + M^{\perp})$ & $p \neq 0$.
So, $p \perp M$ & $p \perp M^{\perp}$
$\Rightarrow p \in M^{\perp}$ & $p \in M^{\perp\perp}$
$\Rightarrow p = 0$.
which is a contradiction

So, $H = M \oplus M^{\perp}$.

Proof
Now take a function
$P: H \to H$ defined by
$P: M \oplus M^{\perp} \to M \oplus M^{\perp}$
$P(\underbrace{x+y}_{z}) = x$ $\quad (x + y \in M + M^{\perp})$.
Now $P(z_1 + z_2) = P(x_1 + y_1 + x_2 + y_2)$
$= x_1 + x_2 = P(x_1 + y_1) + P(x_2 + y_2)$

$\|P(z)\|^2 = \|x\|^2 \le \|x\|^2 + \|y\|^2$

$= \|z\|^2$

$\|x+y\|^2$

$= \langle x+y, x+y \rangle$

$= \|x\|^2 + \|y\|^2$

$\Rightarrow \quad \|P(z)\| \le \|z\|.$

$\Rightarrow \quad \|P\| \le 1.$

Now take $y \in M$ & $z = \frac{y}{\|y\|}$, $y \neq 0$.

Now $\|P\| \geq \left\| P\left(\frac{y}{\|y\|}\right) \right\| = \left\| \frac{y}{\|y\|} \right\| = 1$

$\Rightarrow \quad \|P\| = 1.$

So, P is ~~per~~ bounded with $\|P\| = 1$

& $P^2 = P$. & range$(P) = M$.

Now ~~so~~ also, ker$(P) = M^{\perp}$.