Tutorial on October 3, 2018 (Malay Mandal)

These notes contain a proof of the Hahn-Banach theorem.

Real functionals vs. Complex functionals

Suppose V is a vector space over $\mathbb{C}$. Then it is also a vector space over $\mathbb{R}$. Let us understand the relationship between the two structures on V.

Lemma: Let X and Y be vector spaces over $\mathbb{C}$ and

$$T : X \longrightarrow Y$$

a real linear transformation, i.e. T is linear transformation from the real vector space X to the real vector space Y. Then T is a complex linear transformation if and only if

$(*)$ ——— $T(ix) = i\,Tx \qquad \forall\, x \in X.$

Proof:

The only non-trivial matter that needs checking is:

$$(*) \Rightarrow T(\alpha x) = \alpha T x \qquad \forall\, \alpha \in \mathbb{C},\ x \in X.$$

So suppose $(*)$ holds. Let $\alpha = a + ib$, $a, b \in \mathbb{R}$. Then

$$\begin{aligned}
T(\alpha x) &= T(ax) + T(ibx) \\
&= aTx + b\,T(ix) && (\text{since } T \text{ is } \mathbb{R}\text{-linear}) \\
&= aTx + ib\,Tx && (\text{since } (*) \text{ holds}) \\
&= \alpha T x.
\end{aligned}$$

q.e.d.

Suppose X is a complex vector space and

$$f : X \longrightarrow \mathbb{C}$$

a linear functional. If $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$ then $u : X \longrightarrow \mathbb{R}$ and $v : X \longrightarrow \mathbb{R}$ are real functionals, i.e., they are real linear and $\mathbb{R}$ valued. This is obvious from the definitions. Actually one can recover f from u (or from v). Here is the statement.

<u>Proposition</u>: Let X be a vector space over $\mathbb{C}$.

(a) If $f : X \to \mathbb{C}$ is a linear functional and $u = \mathrm{Re} f$, then

(†) —— $\quad f(x) = u(x) - i u(ix) \qquad x \in X.$

(b) If $u : X \longrightarrow \mathbb{R}$ is a real linear functional and $f : X \longrightarrow \mathbb{C}$ is defined as in (†), then f is complex linear and $\mathrm{Re} f = u$.

(c) Let X be a normed linear space, $f : X \longrightarrow \mathbb{C}$ a <u>bounded</u> linear functional and $u : X \longrightarrow \mathbb{R}$ as in (a). Then

$$\|f\| = \|u\|$$

<u>Proof</u>

(a) Let $v = \mathrm{Im} f$. Then for $x \in X$

$$\begin{aligned} -v(x) + i u(x) &= i(u(x) + i v(x)) \\ &= i f(x) \\ &= f(ix) \qquad \text{(since } f \text{ is } \mathbb{C}\text{-linear)} \\ &= u(ix) + i v(ix). \end{aligned}$$

It follows that $v(x) = -u(ix)$. Since $f(x) = u(x) + i\,v(x)$, we get $f(x) = u(x) - i\,u(ix)$.

(b) If $u : X \longrightarrow \mathbb{R}$ is $\mathbb{R}$-linear and $f : X \longrightarrow \mathbb{C}$ is defined as in $(*)$, then clearly f is $\mathbb{R}$-linear. By the Lemma, it is enough to show $f(ix) = i f(x)$ for $x \in X$. From $(*)$, if $x \in X$, we have

$$\begin{aligned} f(ix) &= u(ix) - i\,u(-x) \\ &= u(ix) + i\,u(x) \\ &= i\,(u(x) - i\,u(ix)) \\ &= i f(x). \end{aligned}$$

This proves (b)

(c) Since $|u(x)| \le |f(x)|$ $\forall x \in X$ hence $\|u\| \le \|f\|$.

Conversely, let $x \in X$ and pick $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$ and $\alpha f(x) = |f(x)|$.

Thus $|f(x)| = \alpha f(x) = f(\alpha x) = u(\alpha x) \le \|u\| \cdot \|\alpha x\| = \|u\|\,\|x\|$.

Thus $\|f\| \le \|u\|$.

This proves (c). q.e.d.

The Hahn-Banach Theorem

Theorem: Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. Let X be a normed linear space over $\mathbb{K}$, M a subspace of X and $f : M \longrightarrow \mathbb{K}$ a bounded linear functional. Then f can be extended to a bounded linear functional $F : X \longrightarrow \mathbb{K}$ such that $\|F\| = \|f\|$.

(Note: By "extension" we of course mean that $F|_M = f$.)

Historical note: This is really a real vector space result. The extension to $\mathbb{C}$ took a while, one of the possible reasons why Banach confined his book on functional analysis to normed linear spaces over $\mathbb{R}$.

<u>Proof</u>:

We will first assume $\mathbb{K} = \mathbb{R}$. WLOG assume $\|f\| = 1$.

If $M = X$ there is nothing to prove. If not, pick $x_0 \in X \setminus M$ and let M_1 be the linear span of M and $\{x_0\}$. Note that any $z \in M_1$ can be written <u>uniquely</u> as

$$z = x + \lambda x_0 \qquad (x \in M,\ \lambda \in \mathbb{R})$$

For $\alpha \in \mathbb{R}$ define

$$f_\alpha : M_1 \longrightarrow \mathbb{R}$$

by the formula

$$f_\alpha(x + \lambda x_0) = f(x) + \lambda \alpha \qquad (x \in M,\ \lambda \in \mathbb{R})$$

It is clear that f_α is linear and $f_\alpha|_M = f$. We would like to pick α such that $\|f_\alpha\| = \|f\|$, i.e., $\|f_\alpha\| = 1$. For this it is enough to prove that

$$(*) \quad \text{———} \quad |f(x) + \lambda\alpha| \leq \|x + \lambda x_0\| \qquad \forall\, x \in M,\ \lambda \in \mathbb{R}$$

The above is trivially true for $\lambda = 0$ and $\forall\, x \in M$.

So assume $\lambda \neq 0$. Then $(*)$ is is equivalent to

$$\left|f\left(\frac{x}{\lambda}\right) + \alpha\right| \leq \left\|\frac{x}{\lambda} + x_0\right\| \qquad \forall\, x \in M,\ \lambda \in \mathbb{R} \setminus \{0\}.$$

This is clearly equivalent to requiring that α satisfies.

$$|f(x) + \alpha| \leq \|x + x_0\| \qquad \forall\, x \in M.$$

Let $z = -x$. Then the requirement is that α satisfies

$$(**) \quad \text{———} \quad |-f(z) + \alpha| \leq \|z - x_0\| \qquad \forall\, z \in M.$$

Now $(**)$ is equivalent to .

$$-\|z-x_0\| \le -f(z)+\alpha \le \|z-x_0\| \quad \forall\, z\in M.$$

In other words, for α to satisfy $(*)$ is the same as requiring that α satisfy

$$(\dagger) \quad f(z)-\|z-x_0\| \le \alpha \le f(z)+\|z-x_0\| \quad \forall\, z\in M$$

The difficulty is in having α satisfy the inequalities simultaneously for every $z\in M$.

Let $A_z = f(z)-\|z-x_0\|$ and $B_z = f(z)+\|z-x_0\|$, $z\in M$

For $y, z\in M$ we have

$$f(y)-f(z) = f(y-z) \le \|y-z\| \le \|y-x_0\| + \|z-x_0\|.$$

Re-arranging the above we get

$$A_y \le B_z \qquad \forall\, y, z\in M.$$

Hence $\sup_{y\in M} A_y \le \inf_{z\in M} B_z$.

Now pick α (at least one such $\alpha\in\mathbb{R}$ exists) s.t.

$$\sup_{y\in M} A_y \le \alpha \le \inf_{z\in M} B_z.$$

For this α, $A_z \le B_z$ $\forall\, z\in M$, i.e, $(\dagger)$ is satisfied. This means $(*)$ is true for this choice of α, and hence $\|f_\alpha\|=1$.

Now let $\mathcal{A}$ be the collection of pairs (N, g), N a subspace of X containing M, and $g: N\longrightarrow\mathbb{R}$ a linear functional with $\|g\|=1$ and s.t. $g|_M = f$. Define a partial order on $\mathcal{A}$ by

$$(N_1, g_1) \prec (N_2, g_2) \quad \text{if} \quad N_1\subset N_2 \text{ and } g_1 = g_2|_{N_1}.$$

If A is a totally ordered set and $\{(N_\alpha, g_\alpha) \mid \alpha\in A\}$ is a chain in $\mathcal{S}$ then it is clear that if $N = \bigcup N_\alpha$, and $g : N \longrightarrow \mathbb{R}$ the unique map s.t. $g|_{N_\alpha} = g_\alpha$, then (N, g) is least upper bound for $\{(N_\alpha, g_\alpha)\}$. Hence by Zorn's Lemma $\mathcal{S}$ has a maximal element $(\tilde{M}, \tilde{f})$. If $\tilde{M} \neq X$, pick $z_0 \in X - \tilde{M}$ and set $\tilde{M}_1$ equal to the linear span of $\tilde{M}$ and z_0. Then by previous argument $\exists$ a linear functional $\Phi : \tilde{M}_1 \longrightarrow \mathbb{R}$ such that $\Phi|_{\tilde{M}} = \tilde{f}$ and $\|\Phi\| = 1$. This contradicts the maximality of $(\tilde{M}, \tilde{f})$. Hence $\tilde{M} = X$. Set $F = \tilde{f}$. This proves the Hahn-Banach Theorem for $\mathbb{K} = \mathbb{R}$.

If $\mathbb{K} = \mathbb{C}$, then let $u = \operatorname{Re} f$. From the Proposition, $\|u\| = \|f\|$. By Hahn-Banach for $\mathbb{K} = \mathbb{R}$, u has a $\mathbb{R}$-linear extension $\tilde{u} : X \longrightarrow \mathbb{R}$ s.t. $\|\tilde{u}\| = \|u\|$. Now define

$$F : X \longrightarrow \mathbb{C}$$

by

$$F(x) = \tilde{u}(x) - i\tilde{u}(ix) \qquad x \in X.$$

From our earlier Proposition, F is complex linear, is an extension of f and

$$\|F\| = \|\tilde{u}\| = \|u\| = \|f\|.$$

q.e.d.