Tutorial - 2

Role of Null sets in Measure Theory.

(X, M, μ) is said to be Complete if $A \in M$ & $\mu(A) = 0$ then

$B \in M$ $\forall$ $B \subseteq A$ and $\mu(B) = 0$.

① **Theorem:**

Every measure space can be completed.

Proof:

Define $M^* = \{ E \subseteq X \mid \exists A, B \in M, A \subset E \subset B$ and $\mu(B \setminus A) = 0 \}$

& $\tilde{\mu}(E) = \mu(A)$.

Things to prove:

1. $\tilde{\mu}$ is well defined
2. M^* is a σ- algebra.

let $E \in M^*$ and suppose $A \subset E \subset B$, $A' \subset E \subset B'$ and

$\mu(B \setminus A) = \mu(B' \setminus A') = 0$. $\{(A, B \in M)\}$

$A \setminus A' \subset E \setminus A' \subset B' \setminus A'$

$\Rightarrow \mu(B' \setminus A') = \mu(A \setminus A') = 0$.

Since $\mu(A \setminus A') = 0$ we get $\mu(A) = \mu(A \cap A')$

Similarly $\mu(A') = \mu(A \cap A')$

$\Rightarrow \mu(A) = \mu(A')$

Thus μ is well defined.

observe $\tilde{\mu}$ is an extension of μ

& $M \subseteq M^*$.

So 1. $X \in M^*$.

2. let $E \in M^*$; $A \subset E \subset B$ & $\mu(B \setminus A) = 0$

then $B^c \subset E^c \subset A^c$ $\mu(A^c \setminus B^c) = \mu(A^c \cap B) = \mu(B \setminus A) = 0$.

Thus $E^c \in M^*$.

3. let $E_i \in M^*$ $\forall i \in \mathbb{N}$

then $\exists$ A_i, B_i s.t $A_i \subset E_i \subset B_i$ $\forall$ $i \in \mathbb{N}$.

Then $\cup A_i \subset \cup E_i \subset \cup B_i$

and $\cup B_i \setminus \cup A_i \subseteq \cup (B_i \setminus A_i)$

$\mu(\cup(B_i \setminus A_i)) = 0$.

Note:

Once the completion is done, a function defined almost everywhere can be extended to X (as a measurable function) in any arbitrary way.

② **Theorem:**

$\{f_i\}$ is a sequence of complex measurable functions defined a.e on X such that $\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty$.

Then 1. $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges almost every where

2. $f \in L^1(\mu)$.

3. $\int f \, d\mu = \sum_{n \geq 1}^{\infty} \int_X f_n \, d\mu$.

Proof:

let S_i be the set on which f_i is defined. & $\mu(\bigcap_{i=1}^{\infty} S_i^c)$

So on $\bigcap_{i=1}^{\infty} S_i$ each function is defined.

$= \mu(\bigcup_{i=1}^{\infty} S_i^c) = \sum_{i=1}^{\infty} \mu(S_i^c) = 0$.

So define

$\phi(x) = \sum |f_n(x)|$ $\forall$ $x \in S = \bigcap_{i=1}^{\infty} S_i$

By MCT and the linearity of integration

$$\int_S \phi \, d\mu < \infty .$$

$\Rightarrow$ $\phi < \infty$ a.e on S.

let $E = \{ x \in S \mid \phi(x) < \infty \}$

so $E^c = S \setminus E \sqcup S^c$

Both have measure zero so $\mu(E^c) = 0$.

So the series converges absolutely on $x \in E$..

So define $f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \forall \; x \in E$.

$|f(x)| < |\phi(x)|$ on E. So $f \in L^1(\mu)$ on E.

define $g_n = f_1 + f_2 \ldots f_n$.

$|g_n| < |\phi| \quad \forall \; x \in E$.

By DCT.

$$\int_E f \, d\mu = \sum_{n=1}^{\infty} \int_E f_n \, d\mu .$$

Since $\mu(E^c) = 0$

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu .$$

Theorem:

let $f : X \longrightarrow [0, \infty]$ measurable, $E \in M$

$\int_E f \, d\mu = 0 \Rightarrow f = 0$ a.e.

Pf:

$A = \{ x \in E \mid f(x) > 0 \}$

Define $A_n = \{ x \in E \mid f(x) > \frac{1}{n} \}$ $\quad A = \bigcup A_n$.

$$\frac{1}{n} \mu(A_n) \leq \int_{A_n} f \, d\mu \leq \int_E f \, d\mu = 0$$

$\Rightarrow \mu(A_n) = 0$

$\Rightarrow \mu(A) = 0$

Similarly Prove $f \in L^1(\mu)$ $\int_E f\,d\mu = 0 \quad \forall \; E \in M$ then $f = 0$ a.e.

③ Theorem:

let $f \in L^1(\mu)$

and $\left|\int_X f\,d\mu\right| = \int_X |f|\,d\mu$

then there is a constant α s.t $\alpha f = |f|$ a.e on X

Pf:

$$\left|\int_X f\,d\mu\right| = \alpha \int_X f\,d\mu = \int_X \alpha f\,d\mu = \int_X u\,d\mu \leq \int_X |f|\,d\mu$$

where u is the real part of αf.

$|\alpha| = 1$

if the equality holds then

$\int_X u\,d\mu = \int_X |f|\,d\mu \Rightarrow |f| = u$ a.e.

$|\alpha f| = |f| = [Re(\alpha f)]$ a.e

$\Rightarrow \alpha f = |f|$ a.e.

④ Theorem:

let $\mu(X) < \infty$, $f \in L^1(\mu)$, S be a closed set in the complex plane

define $A_E(f) = \frac{1}{\mu(E)} \int_E f\,d\mu$, $\mu(E) > 0$

E & $A_E(f) \in S \quad \forall \; E \in M$.

This is the average of f over

Then $f(x) \in S$ a.e

Proof:

take S^c in $\mathbb{C}$. Since this is open, it can be covered using countably many discs whose closure is also contained in S^c.

Call one such disc Δ.

$\Delta = \overline{B(\alpha; r)}$

$E = f^{-1}(\Delta)$

it is sufficient to show that $\mu(E) = 0$

if not. then

$$|A_E(f) - \alpha| = \left| \frac{1}{\mu(E)} \int_E f \, d\mu - \frac{1}{\mu(E)} \int_E \alpha \, d\mu \right|$$

$$= \frac{1}{\mu(E)} \left| \int_E f - \alpha \, d\mu \right| \leq \frac{1}{\mu(E)} \int_E |f-\alpha| \, d\mu$$

Since $f(x) \in \Delta \ \forall \ x \in E$.

$$\frac{1}{\mu(E)} \int_E |f-\alpha| \, d\mu \leq r.$$

But this is a contradiction because $|A_E(f) - \alpha| > r$.

5 <u>Theorem</u>

let $\{E_k\}$ be a sequence of measurable sets in X s.t

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty$$

$$A = \{ x \in X \mid x \in E_k \text{ for infinitely many } k \}$$

then $\mu(A) = 0$.

<u>Proof:</u>

define $g(x) = \sum_{k=1}^{\infty} \chi_{E_k}(x)$

$g(x) = \infty$ if $x \in A$.

$$\int_X g(x) = \sum_{k=1}^{\infty} \mu(E_k) < \infty \quad (\text{By MCT \& linearity})$$

So $g(x) < \infty$ a.e on X.

Thus $\mu(A) = 0$.