Sep 6, 2018 Lecture 9

Let X be locally compact and Hausdorff. A Borel measure μ on X is called inner regular if $\mu(E) = \sup\{\mu(K) \mid K \subset E,\ K \text{ compact}\}$ for every $E \in \mathcal{B}(X)$ and it is called outer regular if $\mu(E) = \inf\{\mu(V) \mid E \subset V,\ V \text{ open}\}$ for every $E \in \mathcal{B}(X)$. A Borel measure which is both inner regular and outer regular is called regular. X is said to be σ-compact if it is the countable union of compact sets.

Throughout X is locally compact and Hausdorff.

Theorem: Suppose X is σ-compact, $\mathcal{M}$ a σ-alg on X, μ a measure on $\mathcal{M}$ s.t., $(\mathcal{M}, \mu)$ satisfy all the properties in the Riesz rep'n theorem. Then

(a) Let $E \in \mathcal{M}$. Given $\varepsilon > 0$ $\exists$ V, F, $F \subset E \subset V$, F closed, V open such that $\mu(V - F) < \varepsilon$

(b) μ is a regular Borel measure

(c) Given $E \in \mathcal{M}$, $\exists$ A, B, $A \subset E \subset B$, A F_σ, B G_δ such that $\mu(A - B) = 0$.

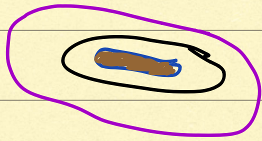
Proof:

(a) Let $X = \bigcup_{n=1}^{\infty} K_n$ with each K_n compact. For each $n \in \mathbb{N}$, $\mu(E \cap K_n)$ is finite and one can find $V_n \supset E \cap K_n$, V_n open, such that $\mu(V_n) \leq \mu(E \cap K_n) < \frac{\varepsilon}{2^n}$. Let $V = \bigcup_n V_n$

$$V - E \subset \bigcup_n (V_n - (E \cap K_n)) \quad \text{and hence } \mu(V - E) < \varepsilon$$

Applying this to E^c we get $W \supset E^c$, W open, s.t. $\mu(W - E^c) < \varepsilon$.

Let $F = W^c$. Then $F \subset E$. Moreover $E - F = W - E^c$ and



hence $\mu(V-F) = \mu(V-E) + \mu(E-F)$

$\leq \mu(V-E) + \mu(W-E^c) \leq 2\varepsilon.$

This proves (a)

(b) Let $H_n = \bigcup_{i=1}^{n} K_i$, $n \in \mathbb{N}$. Then H_n is compact and $H_n \uparrow X$. Let $E \in \mathfrak{M}$. We wish to show that $\mu(E) = \sup\{\mu(K) \mid K \subset E, K \text{ compact}\}$. The non-trivial case is the case $\mu(E) = \infty$. From (a), we can find a closed subset $F \subset E$ s.t. $\mu(F) = \infty$. Now $\{H_n \cap F\}$ is an increasing sequence in $\mathfrak{M}$, with $\bigcup_n H_n \cap F = F$. It follows that given $M > 0$, $\exists\ N \in \mathbb{N}$ s.t. $\mu(H_n \cap F) > M\ \forall n \geq N$. Since $H_n \cap F$ is compact for each n, we are done.

(c) Let $E \in \mathfrak{M}$. For $n \in \mathbb{N}$, pick V_n open, F_n closed, $V_n \supset E \supset F_n$, $\mu(V_n - F_n) < \frac{1}{n}$. Let $A = \bigcup F_n$, $B = \bigcap V_n$. Then A is F_σ and B is G_δ. For each n, $B - A \subset V_n - F_n$, and hence $\mu(B-A) \leq \frac{1}{n}\ \forall n \in \mathbb{N}$. This proves (c)

<u>Theorem</u>: Suppose every open subset of X is σ-compact and μ is Borel measure s.t. $\mu(K) < \infty\ \forall$ compact K. Then μ is regular.

<u>Proof</u>:

We have a positive functional $\Lambda : C_c(X) \to \mathbb{C}$ given by

$$\Lambda f = \int_X f \, d\mu, \quad f \in C_c(X).$$

Recall $|f| \leq \max|f| \cdot \chi_{\text{supp} f}$, and hence $\int_X |f| \, d\mu \leq \max|f| \cdot \mu(\text{supp} f) < \infty$. By Riesz's repn, $\exists$ a measure σ, necessarily regular from the previous results, such that $\int_X f \, d\mu = \int_X f \, d\sigma\ \forall f \in C_c(X)$.

Let V be open. Can find $K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots$, K_i compact s.t. $\bigcup_n K_n = V$, since every open set is σ-compact.

By Urysohn $\exists$ f_n, $K_n \prec f_n \prec V$. Let $g_n = \max(f_1, \ldots, f_n)$. Then $g_n \in C_c(X)$ and $g_n \uparrow \chi_V$. By MCT, we have

$$\mu(V) = \lim \int_X g_n \, d\mu = \lim \int_X g_n \, d\sigma = \sigma(V).$$

Thus $\mu = \sigma$ on open sets. Next let K be compact. Then $V = K^c$ is open and hence σ-compact. In particular we can find compact sets

$$H_1 \subset H_2 \subset \cdots \subset H_n \subset \cdots$$

such that $\bigcup_n H_n = V$. If $V_n = H_n^c$, then $\{V_n\}$ is a decreasing sequence of open sets such that $\bigcap_n V_n = K$. We can find, for each $n \in \mathbb{N}$, $g_n \in C_c(X)$ s.t. $K \prec g_n \prec V_n$. Set $f_n = \min\{g_1, \ldots, g_n\}$. Then $f_n \in C_c(X)$, and $f_n \downarrow \chi_K$ as $n \to \infty$. Now $\int_X f_n \, d\mu = \int_X f_n \, d\sigma < \infty$ $\forall n$, and since $\{f_n\}$ is decreasing DCT applies to both sequences of integrals and we get $\int_X \chi_K \, d\mu = \int_X \chi_K \, d\sigma$, i.e, $\mu(K) = \sigma(K)$.

Thus $\sigma = \mu$ on open sets and compact sets. It follows that $\sigma(E) = \mu(E)$ for every σ-compact set E, for such an E can be written as $E = \bigcup_n H_n$, H_n compact, $H_n \subset H_{n+1}$ $\forall n$. In particular $\sigma(F) = \mu(F)$ for every closed set F (closed sets are σ-compact for X is σ-compact). From here it is clear that

$$\mu(A) = \sigma(A) \qquad \forall\, F_\sigma\text{-sets } A \subset X,$$

for F_σ-sets can be written as increasing unions of countable closed sets.

Let $E \in \mathcal{B}$. If $V \supset E$, E open, then $\mu(E) \leq \mu(V) = \sigma(V)$, whence

$$(*) \qquad \mu(E) \leq \sigma(E)$$

since σ is outer regular (apply inf over V to $\mu(E) \leq \sigma(V)$).

By the last theorem we have an F_σ-set A, $A \subset E$, s.t. $\sigma(A) = \sigma(E)$. This yields

$$\sigma(E) = \sigma(A) \underset{\text{since } A \text{ is } F_\sigma}{=} \mu(A) \leq \mu(E) \underset{\text{via } (*)}{\leq} \sigma(E)$$

Thus $\mu(E) = \sigma(E)$. This means $\mu = \sigma|_{\mathcal{B}(X)}$. Since $\sigma|_{\mathcal{B}(X)}$ is regular (by the previous Theorem), μ is regular. q.e.d.

<u>Obvious consequences</u>: Throughout X is locally compact and Hausdorff.

1. Suppose μ, σ are inner regular on X and $\mu(K) = \mu(K)$ for every compact K. Then $\mu = \sigma$. (Recall that by defn., inner or outer regularity is for Borel measures.)
2. Suppose μ, σ are outer regular on X and $\mu(V) = \sigma(V)$ for every open V. Then $\mu = \sigma$.
3. Suppose every open set in X is σ-compact and σ, μ are Borel measures on X such that $\sigma(K) = \mu(K) < \infty$ for every compact K. Then $\sigma = \mu$.
4. Suppose every open subset of X is σ-compact and we have $\Omega \subset \mathcal{B}(X)$ such that

   (a) For every compact K, $\exists\, Q \in \Omega$ s.t. $K \subset Q$

   (b) Every open V can be written as a countable disjoint union of members of Ω.

If σ, μ are Borel measures on X such that

$$\mu(Q) = \sigma(Q) < \infty \qquad \forall Q \in \Omega$$

then σ and μ are regular and $\sigma = \mu$. We point out that $\mu(K)$ and $\sigma(K)$ are finite for compact K by (a) and (b) above.

Example of 4. above: For $a = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ and $\delta > 0$ define

$$Q(a, \delta) = (\alpha_1, \alpha_1 + \delta] \times \dots \times (\alpha_n, \alpha_n + \delta_n].$$

$Q(a,\delta)$ is called a δ-box with corner a. $Q(a,\delta)$ is clearly Borel.

For $r \in \mathbb{N}$, let P_r be the set of points in $\mathbb{R}^n$ with coordinates which are integral multiples of $1/2^r$. Let Ω_r be the collection of $\frac{1}{2^r}$-boxes with corners at P_r. Then

(a) Ω_r is a partition of $\mathbb{R}^n$ for every r.

(b) If $r \geq s > 0$, and Q, Q' are boxes with $Q \in \Omega_r$, $Q' \in \Omega_s$, then either $Q \subset Q'$ or $Q \cap Q' = \phi$.

(c) If E is a bounded set (in particular, if E is compact), then for each $r \in \mathbb{N}$, E can be covered by a finite disjoint union of members of Ω_r. Indeed, we can find $a \in \mathbb{Z}^n$ and $k \in \mathbb{N}$ s.t. $E \subset Q(a, k)$. Now $Q(a,k)$ is the disjoint union of $(k \cdot 2^r)^n$ members of Ω_r.

(d) Let $\Omega = \Omega_1 \cup \Omega_2 \cup \dots$. Then every open set in $\mathbb{R}^n$ is a disjoint union of members of Ω. (Since Ω is countable this is a countable disjoint union.) This is seen as follows. Let V be an open set in $\mathbb{R}^n$ and x a point in V. Then $\exists\, \varepsilon > 0$

such that $B(x,\varepsilon) \subset V$. For each r, x lies in exactly one member of Ω_r, say $Q_r(x)$. Pick r so that $\sqrt{n}/2^r < \varepsilon$. Now if $a, b \in Q_r(x)$, then $|a-b| \leq \sqrt{n}/2^r < \varepsilon$, whence $Q_r(x) \subset B(x,\varepsilon) \subset V$. Thus V V is the union of all members of Ω lying entirely within V.

Let A_1 be the collection of those members of Ω_1 which lie entirely in V. From $\Omega_2, \Omega_3, \ldots$ remove those boxes which lie inside some box in A_1. From what remains pick all boxes in Ω_2 lying entirely within V. Call this A_2. From $\Omega_3, \Omega_4, \ldots$ remove those boxes which lie in any of the boxes in A_1 or A_2. From what remains pick all boxes in Ω_3 lying entirely in V. Call this collection A_3. Proceeding this way we have $A_1, A_2, A_3, A_4, \ldots$, subsets of Ω. If $A = \bigcup_n A_n$, then clearly $V = \bigcup_{Q \in A} Q$, and $A \subset \Omega$.

Proposition: Let μ, σ be Borel measures on $\mathbb{R}^n$ such that

$$\mu(Q) = \sigma(Q) < \infty \qquad \forall\, Q \in \Omega.$$

Then $\mu = \sigma$.

Proof: First note that every open set in $\mathbb{R}^n$ can be written as the union of closed balls within it whose radii are rational and whose centres have rational coordinates. Thus every open set in $\mathbb{R}^n$ is σ-compact. Using 4 above and the example above, we conclude that $\sigma = \mu$. q.e.d