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Lecture 8

As before X is locally compact T_2, $\Lambda: C_c(X) \longrightarrow \mathbb{C}$ a positive linear functional. As usual V's denote opens, K's compact. If nothing is said about V and K, this will be the underlying assumption. We defined

$$\mu : P(X) \longrightarrow [0,\infty]$$

as $\mu(V) = \sup\{\Lambda f \mid f \prec V\}$ for open V and for $E \subset X$ by

$$\mu(E) = \inf\{\mu(V) \mid E \subset V,\ V \text{ open}\}.$$

We also defined $\mathcal{M}_F = \{E \in P(X) \mid \mu(E) < \infty \text{ and } \mu(E) = \sup\{\mu(K) \mid K \subset E\}\}$; $\mathcal{M} := \{E \in P(X) \mid E \cap K \in \mathcal{M}_F\ \forall K\}$.

The following properties have been proven: I. $\mu(\bigcup_n E_n) \le \sum_n \mu(E_n)$; II. (a) $\mu(K) < \infty$; (b) $\mu(K) = \inf\{\Lambda f \mid K \prec f\}$; III. $\mu(V) = \sup\{\mu(K) \mid K \subset V\}$; IV. If $E = \bigcup_n E_n$, $\{E_n\}$ pairwise disjoint, $E_n \in \mathcal{M}_F$ $\forall n \in \mathbb{N}$, then $\mu(E) = \sum_n \mu(E_n)$, and further if $\mu(E) < \infty$, then $E \in \mathcal{M}_F$ too.

It turns out, and we will see this soon, $\mathcal{M}$ is a σ-algebra containing $\mathcal{B}(X)$ and μ is a measure on $\mathcal{M}$.

Assuming this we showed that

$$\Lambda f = \int_X f\, d\mu, \qquad f \in C_c(X).$$

Note that II and III show that $K \in \mathcal{M}_F$, $\forall K$, and $V \in \mathcal{M}_F$ if and only if $\mu(V) < \infty$. It turns out, as we will see, $\mathcal{M}$ is complete w.r.t. μ and $\mathcal{M}_F = \{E \in \mathcal{M} \mid \mu(E) < \infty\}$.

We now move towards proving that $\mathcal{M}$ is a σ-algebra and μ a measure on it, and other unproved assertions made above and more.

V. If $E \in \mathcal{M}_F$ and $\varepsilon > 0$, there exists an open V and a compact K such that $K \subset E \subset V$ and $\mu(V - K) < \varepsilon$.

Proof:

We can find K and V such that $\mu(V) < \mu(E) + \frac{\varepsilon}{2}$ and $\mu(E) < \mu(K) + \frac{\varepsilon}{2}$. Since $V-K$ is open, and $\mu(V-K) \le \mu(V) < \mu(E) + \frac{\varepsilon}{2} < \infty$, therefore by III, $V - K \in \mathcal{M}_F$. By II, $K \in \mathcal{M}_F$. Thus by IV

$$\mu(V) = \mu(V-K) + \mu(K)$$

hence

$$\begin{aligned}\mu(V-K) &= \mu(V) - \mu(K) \\ &< \mu(E) + \frac{\varepsilon}{2} - \left(\mu(E) - \frac{\varepsilon}{2}\right) \\ &= \varepsilon.\end{aligned}$$

q.e.d.

VI. $A, B \in \mathcal{M}_F \Rightarrow A - B,\ A \cup B,\ A \cap B$ belong to $\mathcal{M}_F$.

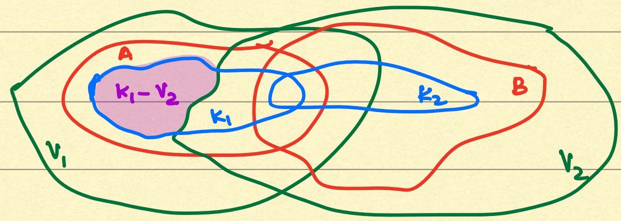
Proof:

Choose $\varepsilon > 0$. We can find opens V_1 and V_2, compacts K_1 and K_2, such that $K_1 \subset A \subset V_1$, $K_2 \subset B \subset V_2$, and $\mu(V_i - K_i) < \varepsilon$ for $i = 1, 2$. We have the following chain of inclusions

$$A - B \subset V_1 - K_2 \subset (V_1 - K_1) \cup (K_1 - V_2) \cup (V_2 - K_2)$$

Now $K_1 - V_2 \subset A - B$ and $K_1 - V_2$ is compact. The above set of inclusions and unions show (use I)

$$\mu(A-B) \le \mu(V - K_1) + \mu(K_1 - V_2) + \mu(V_2 - K_2)$$



i.e.,

$\mu(A-B) \leq 2\varepsilon + \mu(K_1 - V_2)$. Since $K_1 - V_2$ is a compact subset of $A-B$ we see that $A-B$ satisfies (+). On the other hand $\mu(A-B) \leq \mu(A) < \infty$. Hence $A - B \in \mathfrak{M}_F$.

Now $A \cup B$ is the disjoint union of $A-B$ and B, and $\mu(A\cup B) \leq \mu(A) + \mu(B) < \infty$ (by I), and hence by $\overline{\text{IV}}$, $A \cup B$ is in $\mathfrak{M}_F$ since $A-B$ and B are.

Finally, $A \cap B = A - (A-B)$. It follows that $A \cap B \in \mathfrak{M}_F$.

q.e.d.

$\overline{\text{VII}}$. $\mathfrak{M}$ is a σ-algebra which contains $\mathcal{B}(X)$.

Proof:

Recall $\mathfrak{M} = \{E \mid E \subset X,\ E \cap K \in \mathfrak{M}_F \ \forall \text{ compact } K\}$. Since $K \in \mathfrak{M}_F$ for K compact by II, it follows that $X \in \mathfrak{M}_F$.

Let K be an arbitrary compact set in X.

If $A \in \mathfrak{M}$, $A^c \cap K = K - (A \cap K)$, and the latter is in $\mathfrak{M}_F$ by $\overline{\text{VI}}$.

Suppose $\{A_n\}_{n=1}^{\infty}$ is a sequence of members of $\mathfrak{M}$, and let $A = \bigcup_n A_n$. Set $B_1 = A_1 \cap K$ and

$$B_n = (A_n \cap K) - (B_1 \cup \ldots \cup B_{n-1}).$$

By induction each $B_n \in \mathfrak{M}_F$ (use $\overline{\text{VI}}$). The B_n are thus disjoint members of $\mathfrak{M}_F$ whose union is $A \cap K$. Since $\mu(A\cap K) \leq \mu(K) < \infty$, so $A \cap K \in \mathfrak{M}_F$ by $\overline{\text{IV}}$. Thus $A \in \mathfrak{M}$.

Thus $\mathcal{M}$ is a σ-algebra. If C is closed then $C \cap K$ is compact and hence in $\mathcal{M}_F$ by II. Hence every closed set is in $\mathcal{M}$. It follows that every open set is in $\mathcal{M}$, and hence $\mathcal{M} \supset \mathcal{B}(X)$. q.e.d.

VIII. $\mathcal{M}_F = \{E \in \mathcal{M} \mid \mu(E) < \infty\}$.

Proof:

If $E \in \mathcal{M}_F$ then $E \cap K \in \mathcal{M}_F$ $\forall$ compact K since $K \in \mathcal{M}_F$ by II and $\mathcal{M}_F$ is closed under pairwise intersection by VI. Thus $E \in \mathcal{M}$, and certainly $\mu(E) < \infty$.

Conversely, suppose $E \in \mathcal{M}$ and $\mu(E) < \infty$. Since $\mu(E) < \infty$, there exists an open V, $V \supset E$, with $\mu(V) < \infty$. Choose $\varepsilon > 0$. By III we can find a compact subset K of V such that

$$\mu(V) < \mu(K) + \varepsilon$$

Since $\mu(V) < \infty$ and V is open, so by III, $V \in \mathcal{M}_F$. K clearly belongs to $\mathcal{M}_F$. Thus by IV, $\mu(V - K) = \mu(V) - \mu(K) < \varepsilon$.

Now $E \cap K \in \mathcal{M}_F$ since $E \in \mathcal{M}$. Hence there exists a compact subset H of $E \cap K$ such that $\mu(E \cap K) < \mu(H) + \varepsilon$. Now

$$E \subset (V - K) \cup (E \cap K)$$

whence

$$\begin{aligned}\mu(E) &\le \mu(V - K) + \mu(E \cap K) \\ &< \varepsilon + \mu(H) + \varepsilon \\ &= \mu(H) + 2\varepsilon.\end{aligned}$$

Since $H \subset E$, this shows $E \in \mathcal{M}_F$.

$\overline{\text{IX}}$. μ is a measure on $\mathfrak{M}$

Proof:

Let $E_1, E_2, \ldots, E_n, \ldots$ be pairwise disjoint members of $\mathfrak{M}$, and $E = \bigcup_{n=1}^{\infty} E_n$. If $\mu(E) = \infty$, then by I, $\mu(E) = \sum_n \mu(E_n)$, since both sides equal ∞. So assume $\mu(E) < \infty$. Then $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$. By $\overline{\text{VIII}}$, this means $E_n \in \mathfrak{M}_F$ for every n. By $\overline{\text{IV}}$ this means $\mu(E) = \sum_n \mu(E_n)$.

q.e.d.

We have therefore proven the following theorem.

Theorem (The Riesz Representation Theorem):

Let X be a locally compact Hausdorff space, and let Λ be a positive linear functional on $C_c(X)$. Then there exists a σ-algebra $\mathfrak{M}$ in X which contains all Borel sets in X, and there exists a unique positive measure μ on $\mathfrak{M}$ which represents Λ in the sense that

(a) $\Lambda f = \int_X f \, d\mu$ for every $f \in C_c(X)$

and which has the following additional properties:

(b) $\mu(K) < \infty$ for every compact set $K \subset X$.

(c) For every $E \in \mathfrak{M}$, we have

$$\mu(E) = \inf \{\mu(V) \mid E \subset V, V \text{ open}\}.$$

(d) The relation

$$\mu(E) = \sup \{\mu(K) \mid K \subset E, K \text{ compact}\}$$

holds for every open set E, and for every $E \in \mathfrak{M}$ with $\mu(E) < \infty$.

(e) If $E \in \mathfrak{M}$, $A \subset E$, and $\mu(E) = 0$, then $A \in \mathfrak{M}$.

Proof: The uniqueness of μ satisfying (a)–(d) is clear from Problem 1(b), HW 3. Indeed if σ is another measure satisfying (a)–(d), then according to Problem 1(b), HW 3, $\mu(K) = \sigma(K)$ for all compact sets K. By (c) and (d), $\mu(E) = \sigma(E)$ $\forall E \in \mathfrak{M}$. Properties (a)–(d) have been proven. Property (e) is obvious since such an A must lie in $\mathfrak{M}_F$ (by defn of $\mathfrak{M}_F$) and $\mathfrak{M}_F \subset \mathfrak{M}$.

q.e.d.

Remark: Note that (e) says μ is complete. Here are some remarks to help you integrate what is above with what you may see in related courses (e.g., the Probabilistic Measure Theory course). First note that our μ is defined on $\mathcal{P}(X)$, but may not be a measure there. It is so on $\mathfrak{M}$ and hence on $\mathcal{B}(X)$. For $S \subset X$, if we define $\mu^*(S) = \inf\{\mu(E) \mid E \supset S\}$, then by (c) of the Theorem, $\mu^*(S) = \inf\{\mu(V) \mid V \supset S, V \text{ open}\}$. This means $\mu^* = \mu$ on $\mathcal{P}(X)$, i.e., μ on $\mathcal{P}(X)$ is the outer measure of μ on $\mathfrak{M}$. By (e) it is clear from standard results that the completion on $(\mathfrak{M}, \mu)$ is again $(\mathfrak{M}, \mu)$. One way to look at this is to say that $(\mathfrak{M}, \mu)$ is the completion of $(\mathcal{B}(X), \mu|_{\mathcal{B}(X)})$.

The Lebesgue Measure on $\mathbb{R}^n$:

If $f \in C_c(\mathbb{R}^n)$ then $\operatorname{supp} f$ is contained in a closed bounded box $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and one can define

$$\Lambda f = \int_Q f(t_1, \ldots, t_n)\, dt_1\, dt_2 \cdots dt_n$$

where the right side is the Riemann integral. This is well defined and positive. The resulting σ-alg & measure are the Lebesgue σ-alg & measure.