Aug 30, 2018

Lecture 7

As before X is a locally compact Hausdorff space, $\Lambda : C_c(X) \longrightarrow \mathbb{C}$ a positive linear functional. V's denote open sets, K's compact

Summary of main points of last lecture:

A function $\mu : \mathcal{P}(X) \longrightarrow [0,\infty]$ was defined as follows:

For open sets: $\mu(V) = \sup\{\Lambda f \mid f \prec V\}$

For arbitrary sets: $\mu(E) = \inf\{\mu(V) \mid E \subset V\}$.

Three important results we proved were

I. $\mu(\bigcup_n E_n) \leq \sum_n \mu(E_n)$

II. (a) $\mu(K) < \infty$

(b) $\mu(K) = \inf\{\Lambda f \mid K \prec f\}$

III. $\mu(V) = \sup\{\mu(K) \mid K \subset V\}$.

Recall, K denotes compact, V open, E arbitrary.

We also defined $\mathfrak{M}_F$ to be all $E \subset X$ s.t. $\mu(E) < \infty$ and

$$\mu(E) = \sup\{\mu(K) \mid K \subset V\} \quad \text{———} \quad (\dagger).$$

III shows that every open V satisfies $(\dagger)$. So $V \in \mathfrak{M}_F \Leftrightarrow \mu(V) < \infty$. Compact K trivially satisfy $(\dagger)$. By II (a), $K \in \mathfrak{M}_F$. These are two important classes of sets in $\mathfrak{M}_F$.

The set $\mathfrak{M}$ was defined to be

$$\mathfrak{M} = \{E \mid E \subset X,\ E \cap K \in \mathfrak{M}_F \text{ for every compact } K\}.$$

It is clear that every compact set is in $\mathfrak{M}$. Later we will show the following:

(1) $\mathcal{M}$ is a σ-algebra containing $\mathcal{B}(X)$.

(2) μ is a measure on $\mathcal{M}$.

In fact more is true: $\mathcal{M}$ is a complete σ-algebra (i.e., if $S \subset E$, $E \in \mathcal{M}$ with $\mu(E) = 0$, then $S \in \mathcal{M}$), and $\mathcal{M}_F$ consists of precisely those sets E in $\mathcal{M}$ s.t. $\mu(E) < \infty$.

In order to prove $\Lambda f = \int_X f \, d\mu$, $f \in C_c(X)$, we had observed that it is enough to confine ourselves to real-valued f. To show this, by replacing f by $-f$, we saw that it is enough to show

(*) —— $\boxed{\Lambda f \leq \int_X f \, d\mu \, , \ \forall \ f \text{ in } C_c(X) \text{ which are real-valued}}$

Remark: $f \in C_c(X) \Rightarrow \int_X |f| d\mu < \infty$ since $\mu(K) < \infty$ for K compact and $\operatorname{supp}|f|$ is compact.

Proof of (*):

So let $f \in C_c(X)$, $f(X) \subseteq \mathbb{R}$. Let $K = \operatorname{Supp} f$. Since $f(X)$ is either $f(K)$ or $f(K) \cup \{0\}$, it is clear $f(X)$ is compact. Let $[a,b]$ be a finite interval s.t. $f(X) \subset [a,b]$.

Suppose $\varepsilon > 0$ is given. Pick $y_0, y_1, y_2, \ldots, y_n$ s.t.

$$y_0 < a < y_1 < y_2 < \cdots < y_n = b$$

and

$$y_i - y_{i-1} < \varepsilon \quad , \ i = 1, \ldots, n.$$

Let

$$E_i = f^{-1}((y_{i-1}, y_i]) \cap K \qquad i = 1, \ldots, n$$

This means $\quad y_{i-1} < f(x) \leq y_i, \ x \in E_i, \ i = 1, \ldots, n.$

Note that E_i is Borel since f is continuous. Moreover

$$y_i - \varepsilon < f(x) \quad , \quad x \in E_i, \; i=1,\dots,n \text{ ——— (A)}$$

The sets E_i are disjoint and their union is K. By defn of μ, there are open sets V_i, $V_i \supset E_i$, $i=1,\dots,n$ s.t.

$$\mu(V_i) < \mu(E_i) + \frac{\varepsilon}{n} \qquad i=1,\dots,n$$

and such that (by the continuity of f) $f(x) < y_i + \varepsilon$ for $x \in V_i$.

Now $K \subset V_1 \cup V_2 \cup \dots \cup V_n$ and hence we can find $h_i \prec V_i$, $i=1,\dots,n$, s.t. $h_1 + h_2 + \dots + h_n \equiv 1$ on K. Since $f(x) < y_i + \varepsilon$ on V_i, we have

$$h_i f \le (y_i + \varepsilon) h_i \quad , \quad x \in V_i, \; i=1,\dots,n \text{ ——— (B)}$$

The inequalities (A) and (B) are crucial in establishing (*).

Since $\operatorname{supp} f = K$ and $h_1 + \dots + h_n \equiv 1$ on K, we have $f = \sum_{i=1}^n h_i f$.

This means, in particular, that $K \prec \sum_{i=1}^n h_i$, whence by II (b)

$$\mu(K) \le \sum_{i=1}^n \Lambda h_i \text{ ——— (C)}$$

Note that

$$|a| + y_i > 0 \qquad i=1,\dots,n. \text{ ——— (D)}$$

Thus

$$\Lambda f = \sum_{i=1}^n \Lambda(h_i f)$$

$$\le \sum_{i=1}^n (y_i + \varepsilon) \Lambda h_i \qquad \text{(by (B))}$$

$$= \sum_{i=1} (|a| + y_i + \varepsilon) \Lambda h_i - |a| \sum_{i=1} \Lambda h_i$$

$$\leq \sum_{i=1}^{n} (|a| + y_i + \varepsilon) \Lambda h_i - |a|\mu(K) \qquad (\text{by (C)})$$

$$\leq \sum_{i=1}^{n} (|a| + y_i + \varepsilon) \mu(V_i) - |a|\mu(K) \qquad \left(\begin{array}{l}\text{since } h_i \prec V_i \text{, and} \\ \text{since } |a| + y_i + \varepsilon > 0 \\ \text{by (D)}\end{array}\right)$$

$$\leq \sum_{i=1}^{n} (|a| + y_i + \varepsilon) \left(\mu(E_i) + \frac{\varepsilon}{n}\right) - |a|\mu(K)$$

$$= \cancel{|a| \sum_{i=1}^{n} \mu(E_i)} + |a| \sum_{i=1}^{n} \frac{\varepsilon}{n}$$

$$+ \sum_{i=1}^{n} (y_i + \varepsilon) \left(\mu(E_i) + \frac{\varepsilon}{n}\right) - \cancel{|a|\mu(K)}$$

$$= |a| \cdot \varepsilon + \sum_{i=1}^{n} (y_i + \varepsilon)\left(\mu(E_i) + \frac{\varepsilon}{n}\right) \qquad \left(\text{since } \sum_{i=1}^{n} \mu(E_i) = \mu(K)\right)$$

$$= |a| \cdot \varepsilon + \sum_{i=1}^{n} (y_i - \varepsilon)\left(\mu(E_i) + \frac{\varepsilon}{n}\right) + 2\varepsilon \sum_{i=1}^{n} \left(\mu(E_i) + \frac{\varepsilon}{n}\right)$$

$$= \sum_{i=1}^{n} (y_i - \varepsilon)\mu(E_i) + \sum_{i=1}^{n} (y_i - \varepsilon) \cdot \frac{\varepsilon}{n} + |a| \cdot \varepsilon$$

$$+ 2\varepsilon\mu(K) + 2\varepsilon \sum_{i=1}^{n} \frac{\varepsilon}{n}.$$

$$= \sum_{i=1}^{n} (y_i - \varepsilon)\mu(E_i) + \sum_{i=1}^{n} y_i \cdot \frac{\varepsilon}{n} - \varepsilon^2 + |a| \cdot \varepsilon + 2\varepsilon\mu(K) + 2\varepsilon^2$$

$$\leq \sum_{i=1}^{n} (y_i - \varepsilon)\mu(E_i) + \sum_{i=1}^{n} b \cdot \frac{\varepsilon}{n} + \varepsilon|a| + 2\varepsilon\mu(K) + \varepsilon^2.$$

$$= \int_X \left(\sum_{i=1}^{n} (y_i - \varepsilon)\chi_E\right) d\mu + \varepsilon\left(b + |a| + 2\mu(K) + \varepsilon\right)$$

$$\leq \int_X f \, d\mu + \varepsilon\left(b + |a| + 2\mu(K) + \varepsilon\right) \qquad (\text{by (A)})$$

Since $\varepsilon > 0$ is arbitrary, $(*)$ follows. q.e.d.

It remains to prove various properties of $\mathfrak{M}$ and $\mathfrak{M}_F$. Recall we have proven I, II, and III for μ. Here are other properties.

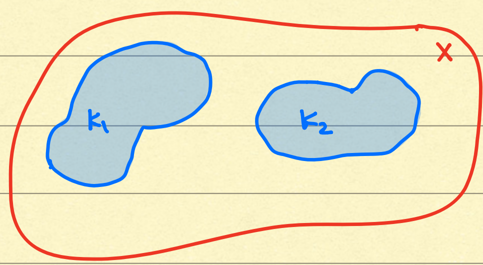
IV. Suppose $E = \bigcup_{n=1}^{\infty} E_n$, where $E_1, E_2, \ldots, E_n, \ldots$ are pairwise disjoint members of $\mathfrak{M}_F$. Then

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E_n).$$

If, further $\mu(E) < \infty$, then $E \in \mathfrak{M}_F$.

Proof:

Let us first prove that if K_1, K_2 are disjoint compact sets then $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$. Setting $V = X \sim K_2$, we see



that $K_1 \subset V$. By Urysohn's lemma we can find $f \in C_c(X)$ such that

$$K_1 \prec f \prec V.$$

Note that $f \equiv 1$ on K_1 and $f \equiv 0$ on K_2.

Let $\varepsilon > 0$ be given. By II we have g such that

$$K_1 \cup K_2 \prec g$$

and

$$\Lambda g < \mu(K_1 \cup K_2) + \varepsilon.$$

Now $K_1 \prec gf$ and $K_2 \prec g(1-f)$. Again by II

$$\mu(K_1) + \mu(K_2) \leq \Lambda(gf) + \Lambda(g(1-f)) = \Lambda g < \mu(K_1 \cup K_2) + \varepsilon.$$

Thus $\mu(K_1) + \mu(K_2) \leq \mu(K_1 + K_2)$. Thus $\mu(K_1) + \mu(K_2) = \mu(K_1 + K_2)$ by I.

IV is clearly true (via I) if $\mu(E) = \infty$. So assume

$\mu(E) < \infty$. Choose $\varepsilon > 0$. Since $E_n \in \mathfrak{M}_F$ for $n \in \mathbb{N}$, there exists, for each $n \in \mathbb{N}$, a compact subset H_n of E_n such that

$$\mu(E_n) < \mu(H_n) + \frac{\varepsilon}{2^n}.$$

Set $K_n = H_1 \cup H_2 \cup \cdots \cup H_n$, $n \in \mathbb{N}$. Then K_n is compact. Moreover, since H_n's are pairwise disjoint, from what we've proven

$$\mu(K_n) = \mu(H_1) + \cdots + \mu(H_n), \qquad n \in \mathbb{N}.$$

Now,

$$\mu(E) \geq \mu(K_n) = \sum_{i=1}^{n} \mu(H_i) > \sum_{i=1}^{n} \mu(E_i) - \varepsilon, \qquad n \in \mathbb{N}.$$

It follows that $\mu(E) \geq \sum_{i=1}^{\infty} \mu(E_i) - \varepsilon$, which means $\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$, since $\varepsilon > 0$ was arbitrary. We are also using I.

It remains to show $E \in \mathfrak{M}_F$ if $\mu(E) < \infty$. With the above $\varepsilon > 0$ and the above H_n and K_n, we have seen that $\mu(K_n) > \sum_{i=1}^{n} \mu(E_i) - \varepsilon$, i.e.,

$$\sum_{i=1}^{n} \mu(E_i) - \mu(K_n) < \varepsilon.$$

On the other hand $\exists\, N \in \mathbb{N}$ s.t. $\mu(E) - \sum_{i=1}^{n} \mu(E_i) < \varepsilon$ for $n \geq N$. Thus

$$0 \leq \mu(E) - \mu(K_N) = \left\{\mu(E) - \sum_{i=1}^{N} \mu(E_i)\right\} + \left\{\sum_{i=1}^{N} \mu(E_i) - \mu(K_N)\right\}$$

$$< 2\varepsilon.$$

Since K_N is compact, it follows that $E \in \mathfrak{M}_F$. q.e.d.