August 28, 2018 Lecture 6

Some bits of measure theory will be done in the tutorial by Nageswaran.

Positive functionals on $C_c(X)$: Let X be a locally compact Hausdorff space and as usual (see HW problems) $C_c(X)$ denotes the space of continuous complex-valued functions on X with compact support.

A linear functional

$$\Lambda : C_c(X) \longrightarrow \mathbb{C}$$

is said to be *positive* if $\Lambda f \geq 0$ for f in $C_c(X)$ such that $f \geq 0$.

The standard example of a positive functional on $C_c(X)$ arises from measure theory. If $\mu : \mathcal{B}(X) \longrightarrow [0, \infty]$ is a measure such that $\mu(K) < \infty$ for every compact set K in X, then

$$f \longmapsto \int_X f \, d\mu \quad , \quad f \in C_c(X)$$

defines a positive linear functional on $C_c(X)$. Note that since $f \in C_c(X)$, and $\mu(\mathrm{Supp}\, f) < \infty$, therefore $\int_X |f| \, d\mu < \infty$. This means $f \in L^1(\mu)$, whence $\int_X f \, d\mu$ is a well-defined complex number.

The point of this discussion is to show that the above class of examples of positive linear functionals on $C_c(X)$ are the only examples.

Notations: V, V_i, V_α etc will denote open subsets of X. K, K_i, K_α etc will denote compact subsets of X.

Let $f \in C_c(X)$, V an open subset of X, K a compact subset of K. We use the following notations:

$K \prec f$ means $0 \le f \le 1$ and $f \equiv 1$ on K

$f \prec V$ means $0 \le f \le 1$ and $\operatorname{Supp} f \subset V$

$K \prec f \prec V$ means $0 \le f \le 1$, $f \equiv 1$ on K, and $\operatorname{Supp} f \subset V$.

Urysohn's Lemma says that if $K \subset V$, then $\exists$ $f \in C_c(X)$ s.t. $K \prec f \prec V$. (As usual K compact, V open).

There is another important result - namely the existence of a partition of unity. Suppose K is compact, $V_1, \dots, V_n$ open subsets of X, and $K \subset V_1 \cup \dots \cup V_n$. Then there exist $h_1, \dots, h_n \in C_c(X)$ such that $h_i \prec V_i$, $i = 1, \dots, n$ and $h_1 + \dots + h_n \equiv 1$ on K.

Suppose

$$\Lambda : C_c(X) \longrightarrow \mathbb{C}$$

is a positive functional on X. We want to find a σ-algebra $\mathcal{M} \supset \mathcal{B}(X)$ and a measure $\mu : \mathcal{M} \longrightarrow [0, \infty]$ such that $\Lambda f = \int_X f \, d\mu \quad \forall f \in C_c(X)$. To make μ unique we require a few more properties (e.g. $\mu(K) < \infty$, $\mu(V) = \sup\{\mu(K) \mid K \subseteq V\}$, $\mu(E) = \inf\{\mu(V) \mid E \subseteq V\}$ etc, which we will enumerate later).

Here is how we start the "construction" of μ from Λ. Set

$$\mu(V) = \sup\{\Lambda f \mid f \prec V\} \quad \text{for } V \text{ open in } X.$$

Note that if $V_1 \subset V_2$ then $\mu(V_1) \leq \mu(V_2)$. In particular for V open,

$$\mu(V) = \inf\{\mu(W) \mid V \subset W,\ W \text{ open in } X\}.$$

This being so, there is no resulting inconsistency if for an arbitrary subset E of X we set

$$\mu(E) = \inf\{\mu(V) \mid E \subset V,\ V \text{ open in } X\}.$$

Once again, if $A \subseteq B$, it is obvious that $\mu(A) \leq \mu(B)$.

I. If $\{E_n\}$ is a sequence of subsets of X then

$$\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

Proof: Let V_1, V_2 be open subsets of X. We will first show that $\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2)$. To that end, let $g \in C_c(X)$ be s.t. $g \prec V_1 \cup V_2$. Let $K = \operatorname{Supp} g$.
We then have $h_i \in C_c(X)$, $h_i \prec V_i$, $i=1,2$, such that $h = h_1 + h_2 \equiv 1$ on K. Now $g = gh$ and $gh_i \prec V_i$, $i=1,2$.
Hence we have:

$$\Lambda g = \Lambda gh = \Lambda gh_1 + \Lambda gh_2 \le \mu(V_1) + \mu(V_2)$$

Since g was arbitrary amongst functions s.t. $g \prec V_1 \cup V_2$, by definition of μ on open sets we have

$$\mu(V) \le \mu(V_1) + \mu(V_2). \qquad \text{——— (1)}$$

If $\mu(E_i) = \infty$ for any i, then the inequality in II is trivially true, and so we may assume $\mu(E_i) < \infty$, $i \in \mathbb{N}$. Pick $\varepsilon > 0$. For each $n \in \mathbb{N}$, there exists an open subset V_n of X such that $E_n \subset V_n$ and

$$\mu(V_n) \le \mu(E_n) + \frac{\varepsilon}{2^n}.$$

Let $V = \bigcup_{n=1}^{\infty} V_n$. Let $g \prec V$, and let $K = \operatorname{supp} g$. Since K is compact, $K \subset V_1 \cup \dots \cup V_n$ for some n, whence $g \prec V_1 \cup \dots \cup V_n$. It follows that

$$\begin{aligned} \Lambda g &\le \mu(V_1 \cup \dots \cup V_n) \\ &\le \sum_{i=1}^{n} \mu(V_i) \qquad \text{(by (1))} \\ &\le \sum_{i=1}^{\infty} \mu(E_i) + \varepsilon. \end{aligned}$$

In particular, by defn of $\mu(V)$, we have

$$\mu(V) \le \sum_{n=1}^{\infty} \mu(E_n) + \varepsilon,$$

i.e. $$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} \mu(E_n) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we are done. //

II. Let K be compact. Then

(a) $\mu(K) < \infty$, and

(b) $\mu(K) = \inf\{\Lambda f \mid K \prec f\}$.

Proof: Let f be s.t. $K \prec f$. Let $0 < \alpha < 1$. Set

$$V_\alpha = \{x \in X \mid f(x) > \alpha\} = f^{-1}(\alpha, \infty).$$

Then V_α is open, and $K \subset V_\alpha$. In particular

$$\mu(K) \leq \mu(V_\alpha).$$

In order to estimate the right side, pick $g \prec V_\alpha$. Then $\alpha g < f$ and hence

$$\alpha \Lambda g = \Lambda(\alpha g) \leq \Lambda f.$$

In particular

$$\mu(K) \leq \mu(V_\alpha) = \sup\{\Lambda g \mid g \prec V_\alpha\} \leq \frac{1}{\alpha} \Lambda f.$$

Letting $\alpha \to 1$ we get

$$\mu(K) \leq \Lambda f \qquad \forall f \text{ s.t. } K \prec f. \quad \text{—— (2)}$$

Since $\Lambda f < \infty$ $\forall f$ s.t. $K \prec f$, this gives (a).

To prove (b), suppose $\varepsilon > 0$ is given. We have an open set V, $V \supset K$, such that

$$\mu(V) \leq \mu(K) + \varepsilon.$$

By Urysohn $\exists f$ s.t. $K \prec f \prec V$. Then

$$\begin{aligned}\mu(K) &\leq \Lambda f && \text{(from (2))}\\ &\leq \mu(V) && \text{(by defn of } \mu(V))\\ &\leq \mu(K) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this gives (b).
q.e.d.

III If V is open then

$$\mu(V) = \sup\{\mu(K) \mid K \subset V,\ K \text{ compact}\}.$$

<u>Proof</u>: Pick α such that $\alpha < \mu(V)$. Then $\exists f$ s.t. $f \prec V$ and $\Lambda f > \alpha$. Let $K = \operatorname{Supp} f$. For every open set W s.t. $W \supset K$, we have $f \prec W$, whence $\Lambda f \le \mu(W)$. Taking infimum over W containing K, we get $\Lambda f \le \mu(K)$. Thus $\mu(K) \geqslant \Lambda f > \alpha$ and $K \subset V$. III follows since α was an arbitrary number less than $\mu(V)$. q.e.d.

Before proving more about μ, here are some definitions.

<u>The sets $\mathfrak{M}_F$ and $\mathfrak{M}$</u>: We define $\mathfrak{M}_F$ to be the collection of subsets E of X such that $\mu(E) < \infty$ and s.t.

$$\mu(E) = \sup\{\mu(K) \mid K \subset E,\ K \text{ compact}\} \quad \text{———} \quad (+)$$

By II (a), $K \in \mathfrak{M}_F$ for compact sets K (such K trivially satisfy (+)). By III, if V is open then $V \in \mathfrak{M}_F$ if and only if $\mu(V) < \infty$. To summarize:

(i) $K \in \mathfrak{M}_F$ for every compact K

(ii) $V \in \mathfrak{M}_F$ for every open set V with $\mu(V) < \infty$.

Let

$$\mathfrak{M} = \{E \mid E \subset X, \text{ and } E \cap K \in \mathfrak{M}_F \ \forall \text{ compact } K\}.$$

It turns out that:

(1) $\mathfrak{M}$ is a σ-algebra containing $\mathcal{B}(X)$.

(2) μ is a measure on $\mathfrak{M}$.

We will prove this later. In fact more is true. $\mathfrak{M}$ is <u>complete</u> in the sense that if $S \subset E$, $E \in \mathfrak{M}$, $\mu(E) = 0$, then $S \in \mathfrak{M}$ (and necessarily $\mu(S)=0$. Moreover one can show that $\mathfrak{M}_F = \{ E \in \mathfrak{M} \mid \mu(E) \in \mathfrak{M} \}$.

<u>$\Lambda = \int_X (-)\, d\mu$</u> : Suppose we believe the above statements. We would like to prove that

$$\Lambda f = \int_X f\, d\mu \quad , \quad f \in C_c(X).$$

It is enough to prove this for real-valued f. Our strategy is to prove that

$$\boxed{\Lambda f \le \int_X f\, d\mu \qquad \forall f \in C_c(X),\ f \text{ real-valued.}}$$

This is enough for the following reason: Suppose $f \in C_c(X)$ and is real-valued. Then $(-f)$ is also real-valued and hence

$$\Lambda(-f) \le \int_X (-f)\, d\mu$$

Since Λ and $\int_X (-)\, d\mu$ are both linear, this gives

$$\Lambda f \ge \int_X f\, d\mu$$

which when combined with the inequality in the red box gives $\Lambda f = \int_X f\, d\mu$ for all real-valued f in $C_c(X)$. The idea is to approximate f by specific sorts of simple functions. In greater detail, let $K = \operatorname{Supp} f$, pick $\varepsilon > 0$, and let $[a,b]$ be an interval s.t. $[a,b] \supset f(X)$. For each $\varepsilon > 0$ pick $y_0, y_1, \ldots, y_n \in \mathbb{R}$ s.t. $y_0 < a < y_1 < y_2 < \cdots < y_n = b$ with $y_i - y_{i-1} < \varepsilon$ for $i = 1, \ldots, n$. Set $E_i = f^{-1}((y_{i-1}, y_i]) \cap K$ $i = 1, \ldots, n$. Then one shows $s = \sum_{i=1}^{n} (y_i - \varepsilon)\chi_{E_i} < f$, and that $\Lambda f \le \int_X s\, d\mu + \varepsilon(\text{constant})$. Details will be given in the next lecture.